CLASSIFICATION OF KNOTTED TORI

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Abstract. For a smooth manifold N denote by $E^m(N)$ the set of smooth isotopy classes of smooth embeddings $N \to \mathbb{R}^m$. A description of the set $E^m(S^p \times S^q)$ was known only for $p = q = 0$ or for $p = 0$, $m \neq q+2$ or for $2m \geq 2(p+q) + \max\{p, q\} + 4$ (in terms of homotopy groups of spheres and Stiefel manifolds). For $m \geq 2p + q + 3$ an abelian group structure on $E^m(S^p \times S^q)$ is introduced. This group is described up to an extension problem: *this group and*

$$E^m(D^{p+1} \times S^q) \oplus \ker \lambda_U \oplus E^m(S^{p+q})$$

are associated to the same group for some filtrations of length four. Here $\lambda_U : E \to \pi_q(S^{m-p-q-1})$ is the linking coefficient defined on the subset $E \subset E^m(S^q \sqcup S^{p+q})$ formed by isotopy classes of embeddings whose restriction to each component is unknotted.

This result and its proof have corollaries which, under stronger dimension restrictions, more explicitly describe $E^m(S^p \times S^q)$ in terms of homotopy groups of spheres and Stiefel manifolds. The proof is based on relations between different sets $E^m(N)$, in particular, on a recent exact sequence of M. Skopenkov.

Contents

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1 Introduction and main results

1.1 Statements of main theoretical results

We consider *smooth* manifolds, embeddings and isotopies.[2] For a manifold N let $E^m(N)$ be the set of isotopy classes of embeddings $N \to S^m$.[3] Abelian group structures on $E^m(D^p \times S^q)$ for $m \ge q+3$ and on $E^m(S^p \times S^q)$ for $m \ge 2p+q+3$ are defined analogously to Haefliger.[4] The main result describes $E^m(S^p \times S^q)$ up to an extension problem. For some motivations see §1.4.

Definitions of $[\cdot]$, the 'embedded connected sum' or 'local knotting' action

$$\# : E^m(N) \times E^m(S^n) \to E^m(N),$$

and of $E^m_\#(N)$. By $[\cdot]$ we denoted the isotopy class of an embedding or the homotopy class of a map.

Assume that $m \ge n+2$ and N is a closed connected oriented n-manifold. Represent elements of $E^m(N)$ and of $E^m(S^n)$ by embeddings $f : N \to S^m$ and $g : S^n \to S^m$ whose images are contained in disjoint balls. Join the images of f, g by an arc whose interior misses the images. Let $[f]\#[g]$ be the isotopy class of the *embedded connected sum* of f and g along this arc, for details see §3.3, cf. [Ha66, Theorem 1.7], [Ha66', Theorem 2.4], [Av16, §1].

For $N = S^q \sqcup S^n$ this construction is made for an arc joining $f(S^n)$ to $g(S^n)$.

For $m \ge n+2$ the operation $\#$ is well-defined. [5] Clearly, $\#$ is an action.

Let $E^m_\#(N)$ be the quotient set of $E^m(N)$ by this action and $q_\# : E^m(N) \to E^m_\#(N)$ the quotient map. A group structure on $E^m_\#(S^p \times S^q)$ is well-defined by $q_\# f + q_\# f' := q_\#(f + f')$, $f, f' \in E^m(S^p \times S^q)$, because $(f\#g) + f' = f + (f'\#g) = (f+f)\#g$ by definition of '+' in §2.1.

The following result reduces description of $E^m(S^p \times S^q)$ to description of $E^m(S^{p+q})$ and of $E^m_\#(S^p \times S^q)$, cf. [Sc71], [CS11, end of §1].

Lemma 1.1 (Smoothing Lemma, see proof in §3.3)**.** *For $m \ge 2p+q+3$ we have $E^m(S^p \times S^q) \cong E^m_\#(S^p \times S^q) \oplus E^m(S^{p+q})$.*

The isomorphism is $q_\# \oplus \overline{\sigma}$, where $\overline{\sigma}$ is 'surgery of $S^p \times *$' defined in §3.3. It has the property $(q_\# \oplus \overline{\sigma})(f\#g) = q_\#(f) \oplus (\overline{\sigma}(f) + g)$ for each $f \in E^m(S^p \times S^q)$, $g \in E^m(S^{p+q})$.

For $m \ge n+3$ denote by

• $\lambda = \lambda^m_{q,n} : E^m(S^q \sqcup S^n) \to \pi_q(S^{m-n-1})$ the linking coefficient that is the homotopy class of the first component in the complement to the second component; [6]

• $E^m_U(S^q \sqcup S^n) \subset E^m(S^q \sqcup S^n)$ the subset formed by the isotopy classes of embeddings whose restriction to *each* component is unknotted.

• $K^m_{q,n} := \ker \lambda \cap E^m_U(S^q \sqcup S^n)$.

[2]In this paper 'smooth' means 'C^1-smooth'. Recall that a smooth embedding is 'orthogonal to the boundary'. For each C^∞-manifold N the forgetful map from the set of C^∞-isotopy classes of C^∞-embeddings $N \to \mathbb{R}^m$ to $E^m(N)$ is a 1–1 correspondence. For a (possibly folklore) proof of this result see [Zh16].

[3]For $m \ge n+2$ classifications of embeddings of n-manifolds into S^m and into $\mathbb{R}^m$ are the same [MAH, 1.1]. It is technically more convenient to consider embeddings into S^m instead of $\mathbb{R}^m$.

[4]The sum operation on $E^m(D^p \times S^q)$ is 'connected sum of q-spheres together with normal p-framings' or 'D^p-parametric connected sum'. The sum operation on $E^m(S^p \times S^q)$ is 'S^p-parametric connected sum', cf. [Sk07, Sk10, MAP], [Sk, Theorem 8]. See Group Structure Lemma 2.2, Remark 2.3 on comparison to previous work and Remark 2.4 on the dimension restrictions.

[5]This is proved analogously to the case $X = D^0_+$ of the Standardization Lemma 2.1.b below, because the construction of $\#$ has an analogue for isotopy, cf. §3.2.

[6]See accurate definition in [MAL], [Sk08', §3]. This phrase makes sense even for $q = n$ because $E^m(S^q \sqcup S^n)$ is the set of isotopy classes of links with *numbered* components. Note that there is another linking coefficient $\lambda_- : E^m(S^q \sqcup S^n) \to \pi_n(S^{m-q-1})$ which is not used here, except in §2.5.

Theorem 1.2 (see proof in §2.2)**.** *For $m \geq 2p+q+3$ the group $E^m_\#(S^p \times S^q)$ has a subgroup isomorphic to $K^m_{q,p+q}$ such that the quotient by this subgroup and $E^m(D^{p+1} \times S^q)$ have isomorphic subgroups with isomorphic quotients.*

Theorem 1.2 and its proof have corollaries (§1.2) which, under stronger dimension restrictions, describe $E^m(S^p \times S^q)$ more explicitly, in terms of homotopy groups of spheres and Stiefel manifolds. The subgroups of Theorem 1.2 have natural descriptions (Remark 2.9).

The main predecessor of Theorem 1.2 is essentially known result Corollary 1.8.a. Theorem 1.2 and the Smoothing Lemma 1.1 are new results only for

$$(*) \qquad 1 \leq p < q \quad \text{and} \quad 2m \leq 3q + 2p + 3,$$

by Remark 1.11.a. Analogous remark holds for results of §1.2. There were known weaker versions of the Smoothing Lemma 1.1 [Sk06, Smoothing Theorem 8.1], [CRS12, Proposition 5.6], and rational versions of Theorem 1.2 [CRS12, Sk15].

In the proofs of the Smoothing Lemma 1.1 and Theorem 1.2 the dimension restrictions (*) are not required. So I give new proofs of known cases $2m \geq 3q + 2p + 4$ (in the stronger form of Conjecture 1.3 and Corollary 1.8.a). These new proofs are only interesting for $1 \leq p < q$, by Remark 1.11.a.

Conjecture 1.3. *For $m \geq 2p+q+3$*

$$E^m(S^p \times S^q) \cong E^m(D^{p+1} \times S^q) \oplus K^m_{q,p+q} \oplus E^m(S^{p+q}).$$

Remark 1.4. (a) Conjecture 1.3 is equivalent to $E^m_\#(S^p \times S^q) \cong E^m(D^{p+1} \times S^q) \oplus K^m_{q,p+q}$ by the Smoothing Lemma 1.1.

(b) Conjecture 1.3 is known to be true for $p = 0$ or $p \geq q$ or $2m \geq 3q + 2p + 4$, see Remark 1.11.a. Conjecture 1.3 is true for $(m, p, q) \in \{(11, 1, 6), (17, 1, 10)\}$ by Theorem 1.7.a because $\pi_6(V_{5,2}) \cong \mathbb{Z}_2$ and $\pi_{10}(V_{7,2}) = 0$ [Pa56].

(c) A candidate for an isomorphism from right to left in Conjecture 1.3 is $r \oplus \sigma|_{K^m_{q,p+q}} \oplus {}_{\mathrm{i}}\#$, where r is the restriction map, σ is defined in §2.2 and ${}_{\mathrm{i}}\#$ is defined in §2.1. If $p = 0$, then I can prove that this sum is an isomorphism only for $2m = 3q + 3$ (Theorem 2.13.a), although Conjecture 1.3 is true (Remark 1.11.a). Analogously, proof of Theorem 1.2 does not show that $[r \oplus \sigma|_{K^m_{q,p+q}} \oplus {}_{\mathrm{i}}\#] \otimes \mathbb{Q}$ is an isomorphism, in spite of Corollary 1.6.

(d) *Description of known results for $m \leq 2p+q+2$ and $p, q \geq 1$.* If $p > q$, then $2p+q+2 > 2q+p+2$, so after exchange of p, q the inequality $m \leq 2p+q+2$ remains fulfilled. So it suffices to present description for $p \leq q$. Then $E^m(S^p \times S^q)$ is known only for $(m, p, q) = (6k, 2k-1, 2k)$ [Sk08, Theorem 2.14] or $m = p+q+1$, $p \geq 2$ [LNS]; $E^m_\#(S^p \times S^q)$ is known only for $m \geq \frac{3q}{2} + p + 2$ [Sk08, Theorem 3.9]. Sometimes there are no group structures compatible with natural constructions and invariants, see Remark 2.4. For $(m, p, q) = (7, 1, 3)$ see [CS].

The sign $\circ$ of the composition is often omitted.

Plan of the paper should be clear from the contents of the paper and the following diagram. Subsections are independent on each and other (except maybe for a few references which could be ignored) unless joined by a sequence of arrows.

$$\begin{tikzcd}
1.1 \arrow[rrr] \arrow[d] \arrow[drr] & & & 2.1 \arrow[rrrrr] \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] & & & & & 2.2 \arrow[d] \arrow[dr] \arrow[drr] & & \\
1.2 \arrow[r] & 1.3 & 1.4 & 3.1 & 3.2 & 3.3 & 2.3 \arrow[r] & 2.4 & 3.4 & 3.5 & 2.5
\end{tikzcd}$$

1.2 Corollaries

Corollary 1.5. *For $m \geq 2p+q+3$ we have*

$$|E^m(S^p \times S^q)| = |E^m(D^{p+1} \times S^q)| \cdot |K^m_{q,p+q}| \cdot |E^m(S^{p+q})|$$

(more precisely, whenever one part is finite, the other is finite and they are equal).

Denote by $V_{s,t}$ the Stiefel manifold of t-frames in $\mathbb{R}^s$. Identify $\pi_q(V_{s,1})$ with $\pi_q(S^{s-1})$. The group $\pi_q(V_{m-q,p+1})$ is calculated for many cases, see e.g. [Pa56], [CFS, Lemma 1.12].

Theorem 1.2 and the isomorphism $E^m(D^{p+1}\times S^q)\otimes\mathbb{Q} \cong [\pi_q(V_{m-q,p+1})\oplus E^m(S^q)]\otimes\mathbb{Q}$ [CFS, Lemma 2.15] imply the following rational analogue of Conjecture 1.3.

Corollary 1.6. *For $m \geq 2p+q+3$*

$$E^m(S^p \times S^q) \otimes \mathbb{Q} \cong [\pi_q(V_{m-q,p+1}) \oplus E^m(S^q) \oplus K^m_{q,p+q} \oplus E^m(S^{p+q})] \otimes \mathbb{Q}.$$

The right-hand side groups of Corollary 1.6 are known by Theorem 1.9 below and [CFS, Theorems 1.1, 1.9 and Lemma 1.12]. Thus Corollary 1.6 allows calculation of $\operatorname{rk} E^m(S^p \times S^q)$. This is known [Sk15, Corollary 1.7], so Corollary 1.6 is not a new result. However, our deduction of Corollary 1.6 is interesting because it uses Theorem 1.2 instead of information on the groups involved in [Sk15, §4, proof of Corollary 1.7]; in this sense our deduction explains why the isomorphism holds.

Theorem 1.7 (see proof in §2.2)**.** *(a) If $1 \leq p < l$ and p is not divisible by 4, then $E^{6l-p}_\#(S^p\times S^{4l-p-1}) \cong \mathbb{Z}\oplus G_{l,p}$ for a certain finite group $G_{l,p}$ such that $G_{l,p}$ and $\pi_{4l-p-1}(V_{2l+1,p+1})$ have isomorphic subgroups with isomorphic quotients.*

(b) If $E^m(D^{p+1} \times S^q)$ is finite, $K^m_{q,p+q}$ is free and $m \geq 2p+q+3$, then $E^m_\#(S^p \times S^q) \cong K^m_{q,p+q} \oplus G^m_{p,q}$ for a certain group $G^m_{p,q}$ such that $G^m_{p,q}$ and $E^m(D^{p+1} \times S^q)$ have isomorphic subgroups with isomorphic quotients.

Definition of $\mathbb{Z}_{(s)}$ and the maps pr_k,

$$\tau = \tau^m_{p,q} : \pi_q(V_{m-q,p+1}) \to E^m(D^{p+1} \times S^q).$$

Denote by $\mathbb{Z}_{(s)}$ the group $\mathbb{Z}$ for s even and $\mathbb{Z}_2$ for s odd.

Denote by pr_k the projection of a Cartesian product onto the k-th factor.

Represent an element of $\pi_q(V_{m-q,p+1})$ by a smooth map $x : S^q \to V_{m-q,p+1}$. By the exponential law this map can be considered as a map $x : \mathbb{R}^{p+1} \times S^q \to \mathbb{R}^{m-q}$. The latter map can be normalized to give a map $\widehat{x} : D^{p+1} \times S^q \to D^{m-q}$. Let $\tau[x]$ be the isotopy class of the composition $D^{p+1}\times S^q \overset{\widehat{x}\times\mathrm{pr}_2}{\to} D^{m-q}\times S^q \overset{\mathrm{i}}{\to} S^m$, where i is the standard embedding (see accurate definition in §2.1). [MAK], [Sk08', §6]. Clearly, τ is well-defined and is a homomorphism.

Corollary 1.8. *Assume that $m \geq 2p+q+3$.*

(a) If $2m \geq 2p+3q+4$, then $q_\#\tau r : \pi_q(V_{m-q,p+1}) \to E^m_\#(S^p \times S^q)$ is an isomorphism.

(b) If $2m \geq p+3q+4$, then $E^m_\#(S^p \times S^q)$ and $\pi_q(V_{m-q,p+1})$ have isomorphic subgroups with isomorphic quotients. [7]

[7] The case $2m = 2p+3q+3$ is considered in [Sk06]. Statements [Sk06, Main Theorem 1.4.AD,PL] in the first two arxiv versions is false; [Sk06, Main Theorem 1.3] and [Sk08', Theorem 3.11] are correct. The mistakes are corrected in the present paper except in the case $(m,p,q) = (7,1,3)$ for which see [CS]. The mistakes were in the relation $\tau_p(w_{l,p}) = 2\omega_p$ of [Sk06, the Relation Theorem 2.7, the Almost Smoothing Theorem 2.3].
I conjecture that the groups of (b) are in fact isomorphic. This is a particular case of Conjecture 1.3. This case could hopefully be proved using ideas of [Sk06].

(c) If $2m \ge 3q+4$, then $E^m_\#(S^p\times S^q)$ has a subgroup isomorphic to $\pi_{p+2q+2-m}(V_{M+m-q-1,M})$, whose quotient and $\pi_q(V_{m-q,p+1})$ have isomorphic subgroups with isomorphic quotients.

(d) If $2m = 3q+3$, then $E^m_\#(S^p\times S^q)$ has a subgroup isomorphic to $\pi_{p+2q+2-m}(V_{M+m-q-1,M})$, whose quotient has a subgroup isomorphic to $\mathbb{Z}_{(m-q-1)}$, whose quotient and $\pi_q(V_{m-q,p+1})$ have isomorphic subgroups with isomorphic quotients.

Deduction of Corollary 1.8.a from known results. (See a new direct proof of Corollary 1.8.a in §2.3.) Consider the following diagram

$$\begin{tikzcd} \pi_q(V_{m-q,p+1}) \arrow[r, "\tau"] & E^m(D^{p+1}\times S^q) \arrow[r, "r"] & E^m(S^p\times S^q) \arrow[r, "q_\#"] \arrow[ll, bend right, "\widehat{\alpha}"'] & E^m_\#(S^p\times S^q) \end{tikzcd}$$

Here $\widehat{\alpha}$ is a map such that $\widehat{\alpha}r\tau = \mathrm{id}$ and $\widehat{\alpha}(f\#g) = \widehat{\alpha}(f)$ for each $f\in E^m(S^p\times S^q)$ and $g\in E^m(S^{p+q})$; such a map exists by [Sk02, Torus Lemma 6.1] ($\widehat{\alpha} := \rho^{-1}\sigma^{-1}\,\mathrm{pr}_1\,\gamma\alpha$ in the notation of that lemma). Hence $r\tau$ is injective and $q_\# r\tau$ is injective.

Take any $f\in E^m(S^p\times S^q)$. Let $f' := r\tau\widehat{\alpha}(f)$. Then $\widehat{\alpha}(f') = \widehat{\alpha}(f)$. Then by [Sk02, Corollary 1.6.i] and since the smoothing obstruction assuming values in $E^m(S^{p+q})$ is changed by $g\in E^m(S^{p+q})$ if f is changed to $f\#g$, we obtain $q_\# f = q_\# f' = q_\# r\tau\widehat{\alpha}(f)$. Since $q_\#$ is surjective, we see that $q_\# r\tau$ is surjective. $\square$

Corollaries 1.8.b,c,d are proved at the end of §1.3.

Theorem 1.7.a improves a particular case of Corollary 1.8.c for $2m = 3q+p+3$.

1.3 Calculations

The group $E^m(S^n)$ is calculated for some cases when $m \ge n+3$ [Ha66, Mi72]. In particular,

(S) $E^m(S^n) = 0$ for $2m \ge 3n+4$.

(S') $E^m(S^n) \cong \mathbb{Z}_{(m-n-1)}$ for $2m = 3n+3$.

Theorem 1.9. *For $m-3 \ge q, n$ we have $E^m_U(S^q\sqcup S^n) \cong \pi_q(S^{m-n-1})\oplus K_{q,n}$. [Ha66', Theorem 2.4 and the text before Corollary 10.3]*

The group $K^m_{p,p+q}$ (or, equivalently, $E^m_U(S^q\sqcup S^{p+q})$) is calculated in terms of homotopy groups of spheres and Whitehead products [Ha66', Sk09], [CFS, Theorem 1.9]. In particular,

(L) $K^m_{q,p+q} = 0$ for $2m \ge 3q+p+4$ [Sk08, Theorems 3.1 and 3.6.a]; this also follows by (L').

(L') $K^m_{q,p+q} \cong \pi_{p+2q+2-m}(V_{M+m-q-1,M})$ for $m \ge \frac{2p+4q}{3}+2$ and M large [Sk08, Theorem 3.6] (the isomorphism from the left to the right is defined in [Ha66]).

The group $E^m(D^{p+1}\times S^q)$ can be calculated using Theorem 1.10 below. E.g. by Theorem 1.10, (S), (S') and since for $2m \ge 3q+2$ the normal bundle of any embedding $S^q\to\mathbb{R}^m$ is trivial [Ke59], we have the following.

(D) $\tau : \pi_q(V_{m-q,p+1}) \to E^m(D^{p+1}\times S^q)$ (defined in §1.2) is an isomorphism for $2m \ge 3q+4$.

(D') $E^m(D^{p+1}\times S^q)$ has a subgroup $\mathbb{Z}_{(m-q-1)}$ whose quotient is $\pi_q(V_{m-q,p+1})$ for $2m = 3q+3$.

Theorem 1.10. *For $m \ge q+3$ the following sequence is exact:*

$$\dots \to E^{m+1}(S^{q+1}) \xrightarrow{\xi} \pi_q(V_{m-q,p+1}) \xrightarrow{\tau} E^m(D^{p+1}\times S^q) \xrightarrow{\rho} E^m(S^q) \to \dots$$

Here ρ is the restriction map and ξ is defined below. [CFS, Theorem 2.14], [Sk15, Theorem 2.5], cf. [Ha66, Corollary 5.9].

Definition of a p-framing. A *p-framing* in a vector bundle is a system of p ordered orthogonal normal unit vector fields on the zero section of the bundle.

Definition of the map ξ from Theorem 1.10. Informally, ξ is the obstruction to the existence of a normal $(p+1)$-framing of an embedding. A formal definition is as follows [Sk15, Sketch of proof of Theorem 2.5 in p.7]. Take an embedding $f : S^{q+1} \to S^{m+1}$. Take a normal $(m-q)$-framing of the image $f(D^{q+1}_-)$ of the lower hemisphere and a normal $(p+1)$-framing of the image $f(D^{q+1}_+)$ of the upper hemisphere. These framings are unique up to homotopy. Thus the hemispheres of $f(S^{q+1})$ are equipped with a $(p+1)$-framing and an $(m-q)$-framing. Using the $(m-q)$-framing identify each fiber of the normal bundle to $f(D^{q+1}_-)$ with the space $\mathbb{R}^{m-q}$. Define a map $S^q \to V_{m-q,p+1}$ by mapping point $x \in S^q$ to the $(p+1)$-framing at the point $f(x)$. Let $\xi[f]$ be the homotopy class of this map.

Remark 1.11. (a) Assume that $2m \geq 3p + 3q + 4$. Then the Smoothing Lemma 1.1 holds by (S). Theorem 1.2 and Conjecture 1.3 are true by (S), (D), (L) and the case $2m \geq 3p + 3q + 4$ of Corollary 1.8.a [Sk08, Theorem 3.9].

For $p \geq q$ and $m \geq 2p + q + 3$ we have $2m \geq 3p + 3q + 4$. Hence the Smoothing Lemma 1.1, Theorem 1.2 and Conjecture 1.3 are true.

For $2m \geq 3q + 2p + 4$ the Smoothing Lemma 1.1 is [Sk06, Theorem 1.2.DIFF] (an alternative proof follows from [CRS12, Proposition 5.6] analogously to [Sk08', §4, proof of Higher-dimensional Classification Theorem (a)]). For $2m \geq 3q + 2p + 4$ Theorem 1.2 and Conjecture 1.3 follow from Corollary 1.8.a and (D) (which are easy corollaries of known results).

Assume that $p = 0$. By [Ha66', Theorem 2.4] the Smoothing Lemma 1.1 holds. By [Ha66', Theorem 2.4] and Theorem 1.9 $E^m_\#(S^0 \times S^q) \cong \pi_q(S^{m-q-1}) \oplus E^m(S^q) \oplus K^m_{q,q}$. Since $\lambda^m_{q,q} r \tau^m_{1,q} = \mathrm{id}\, \pi_q(S^{m-q-1})$, by Theorem 1.10 $E^m(D^1 \times S^q) \cong \pi_q(S^{m-q-1}) \oplus E^m(S^q)$. Hence Theorem 1.2 and Conjecture 1.3 are true.

(b) The smallest m for which there are p, q such that $1 \leq p < q$ and $2p + q + 3 \leq m \leq (3q+2p+3)/2$ are $m = 10, 11, 12$. Then $p = 1$ and $q = m-5$. Hence by the Smoothing Lemma 1.1, Theorems 1.2, 1.7.a, Corollaries 1.8.b,c,d and [Pa56, Ha66]

- $|E^{10}(S^1 \times S^5)| = |E^{10}_\#(S^1 \times S^5)| = 4$. Cf. [Sk15, Example 1.4].
- $E^{11}_\#(S^1 \times S^6) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$ and $E^{11}(S^1 \times S^6) \cong \mathbb{Z}_2 \oplus \mathbb{Z} \oplus E^{11}(S^7)$, of which $E^{11}(S^7)$ is rank one infinite.
- $E^{12}_\#(S^1 \times S^7) \cong \mathbb{Z}^2 \oplus G$, where $|G|$ is a divisor of 8, and $E^{12}(S^1 \times S^7) \cong \mathbb{Z}^2 \oplus G \oplus E^{12}(S^8)$, of which $E^{12}(S^8)$ is finite.

Proof of Corollaries 1.8.b,c,d. These results follow from Theorem 1.2 and (D,L), (D,L'), (D',L'), respectively. Here (L') is applicable because $\max\{2p+q+3, \frac{3q+3}{2}\} \geq \frac{2p+4q}{3} + 2$ (indeed, the opposite inequalities imply $4p + 3 < q < 4p + 3$). $\square$

1.4 Some general motivations

This paper is on the classical Knotting Problem: *for an n-manifold N and a number m, describe isotopy classes of embeddings $N \to \mathbb{R}^m$.* For recent surveys see [Sk08', MAH]; whenever possible I refer to these surveys not to original papers.

Many interesting examples of embeddings are embeddings $S^p \times S^q \to \mathbb{R}^m$, i.e. *knotted tori*. See references in [MAK]. Classification of knotted tori is a natural next step after the Haefliger link theory [Ha66'] and the classification of embeddings of highly-connected manifolds [Sk08, §2]. Such a step gives some insight or even precise information concerning embeddings of *arbitrary* manifolds [Sk07, Sk10', Sk], and reveals new interesting relations to algebraic topology.

The Knotting Problem is more accessible for

$$2m \geq 3n + 4,$$

when there are some classical complete readily calculable classifications of embeddings [Sk08', §2, §3], [MAH]. Cf. (S) of §1.3.

The Knotting Problem is much harder for $2m < 3n+4$: if N is a closed manifold that is not a disjoint union of homology spheres, then until recently no complete readily calculable isotopy classification was known. This is in spite of the existence of many interesting approaches including methods of Haefliger-Wu, Browder-Wall and Goodwillie-Weiss [Sk08, §5], [Wa70, GW99, CRS04] (cf. [Sk10, footnote 2]). Analogously, for $2m < 3n + 4$ if N is a closed manifold that is not $[(n-2)/2]$-connected, then until recently no description the set $E^m_\#(N)$ of 'embeddings modulo knots' was known.

Recent results for $2m < 3n + 4$ concern embeddings of d-connected n-manifolds for $2m \geq 3n + 3 - d$ [Sk02, Sk08'], embeddings of 3- and 4-dimensional manifolds [Sk06, Sk08', Sk10, CS11, CS] and rational classification of embeddings $S^p \times S^q \to \mathbb{R}^m$ under stronger dimension restriction than $m \geq 2p + q + 3$ [CRS07, CRS12, CFS] (the latter used results stated in §2.1 of this paper). Methods developed in those papers work hardly for $2m < 3n + 3 - d$, for high-dimensional manifolds and without the stronger dimension restriction.

The new ideas allowing to treat the hardest cases in this paper follow [Sk15] and unpublished work [Sk06]. One idea is to find *relations* between different sets of (isotopy classes of) embeddings, invariants of embeddings and geometric constructions of embeddings. Group structures on sets of embeddings are constructed. Then such relations are formulated in terms of exact sequences. The most non-trivial exact sequence is the main theoretical result of [Sk15] (this is [Sk15, Theorem 1.6], i.e. the exactness of the $\nu\sigma(i\zeta\lambda')$-sequence from the proof of Theorem 1.2 in §2.2, which extends Lemma 2.10.a and which was proved using results stated in §2.1 of this paper). A deeper idea is to find *connections* between such relations, formulated in terms of diagrams involving the exact sequences. These ideas are hopefully interesting in themselves.

2 Proofs of the main results modulo lemmas

2.1 Standardization and group structure

Definition of the inclusion $\mathbb{R}^q \subset \mathbb{R}^m$ and of $\mathbb{R}^m_\pm, D^m_\pm, 0_k, l, T^{p,q}, T^{p,q}_\pm$. For each $q \leq m$ identify the space $\mathbb{R}^q$ with the subspace of $\mathbb{R}^m$ given by the equations $x_{q+1} = x_{q+2} = \dots = x_m = 0$ [Ha66] (the notation in [Ha66', Sk15] is 'the opposite'). Analogously identify D^q, S^q with the subspaces of D^m, S^m.

Define $\mathbb{R}^m_+, \mathbb{R}^m_- \subset \mathbb{R}^m$ and $D^m_+, D^m_- \subset S^m$ by equations $x_1 \geq 0$ and $x_1 \leq 0$, respectively. Then $S^m = D^m_+ \cup D^m_-$. Note that $0 \times S^{m-1} = \partial D^m_+ = \partial D^m_- = D^m_+ \cap D^m_- \neq S^{m-1}$. Denote by 0_k the vector of k zero coordinates,

$$1_k := (1, 0_k), \quad l := m - p - q - 1, \quad T^{p,q} := S^p \times S^q \quad \text{and} \quad T^{p,q}_\pm := D^p_\pm \times S^q.$$

Definition of the standard embedding i. Assume that $m > p + q$. Informally, *the standard embedding* is the smoothing of the composition

$$D^{p+1} \times D^{q+1} \cong D^{q+1} \times D^{p+1} \cong D^{q+1} \times 0_l \times \frac{1}{2} D^{p+1} \hookrightarrow D^{q+1} \times D^l \times D^{p+1} \cong D^{m+1}.$$

Formally, define *the standard embedding* [8]

$$\mathrm{i} = \mathrm{i}_{\,m,p,q} : D^{p+1} \times D^{q+1} \to D^{m+1} \quad \text{by} \quad \mathrm{i}(x, y) := (y\sqrt{2 - |x|^2}, 0_l, x)/\sqrt{2}.$$

[8] The image of $T^{p,q}$ under this embedding is the boundary of a certain neighborhood of $S^q \subset S^m$ in $f(S^{p+q+1})$, where embedding $f : S^{p+q+1} \to S^m$ is defined by $(y, z) \mapsto (y, 0_l, z)$, $y \in \mathbb{R}^{q+1}$.

Note that $\mathrm{i}(D^{p+1}\times S^q)\subset S^m$, $\mathrm{i}(D^{p+1}\times D^q_\pm)\subset D^m_\pm$ and $\mathrm{i}_{m,p,q}$ is the abbreviation [9] of $\mathrm{i}_{m+1,p+1,q}$ but not of $\mathrm{i}_{m+1,p,q+1}$. Denote by the same notation 'i' abbreviations of i (it would be clear from the context, to which sets).

Definition of a standardized map and homotopy. Take a subset $X\subset S^p$. A map $f: X\times S^q\to S^m$ is called *standardized* if

$$f(X\times \mathrm{Int}\, D^q_+)\subset \mathrm{Int}\, D^m_+ \quad \text{and} \quad f|_{X\times D^q_-}=\mathrm{i}_{\,m,p,q}.$$

Cf. [Sk07, Remark after definition of the standard embedding in §2].

A homotopy $F: X\times S^q\times I\to S^m\times I$ is called *standardized* if

$$F(X\times \mathrm{Int}\, D^q_+\times I)\subset \mathrm{Int}\, D^m_+\times I \quad \text{and} \quad F|_{X\times D^q_-\times I}=\mathrm{i}\times \mathrm{id}\, I.$$

Lemma 2.1 (Standardization Lemma; see proof in §3.1)**.** *Let X denote either D^p_+ or S^p. For $X=S^p$ assume that $m\ge 2p+q+3$.*

(a) Each embedding $X\times S^q\to S^m$ is isotopic to a standardized embedding.

(b) If standardized embeddings $X\times S^q\to S^m$ are isotopic, then there is a standardized isotopy between them.

Definition of the reflections R, R_j. Let $R:\mathbb{R}^m\to\mathbb{R}^m$ be the the reflection of $\mathbb{R}^m$ with respect to the hyperplane given by equations $x_1=x_2=0$, i.e., $R(x_1,x_2,x_3,\dots,x_m):=(-x_1,-x_2,x_3,\dots,x_m)$. Let R_j be the reflection of $\mathbb{R}^m$ with respect to the hyperplane $x_j=0$, i.e., $R_j(x_1,x_2,\dots,x_{j-1},x_j,x_{j+1},\dots,x_m):=(x_1,x_2,\dots,x_{j-1},-x_j,x_{j+1},\dots,x_m)$.

Lemma 2.2 (Group Structure Lemma; see proof in §3.2)**.** *Let X denote either D^p_+ or S^p. For $X=D^p_+$ assume that $m\ge q+3$, for $X=S^p$ assume that $m\ge 2p+q+3$. Then a commutative group structure on $E^m(X\times S^q)$ is well-defined by the following construction.*

Take standardized embeddings $f,g: X\times S^q\to S^m$. Let $[f]+[g]$ be the isotopy class of the embedding h_{fg} defined by

$$h_{fg}(x,y):=\begin{cases} f(x,y) & y\in D^q_+ \\ R(g(x,Ry)) & y\in D^q_- \end{cases}.$$

The two formulas agree on $X\times(D^q_+\cap D^q_-)$ because $\mathrm{i}(x,y)=R\,\mathrm{i}(x,Ry)$; clearly, h_{fg} is an embedding.

Let $0:=[\mathrm{i}]$. Let $-[f]:=[\overline{f}]$, where $\overline{f}(x,y):=R_2f(x,R_2y)$.

Define the 'embedded connected sum' or 'local knotting' map

$${}_{\mathrm{i}}\#: E^m(S^{p+q})\to E^m(T^{p,q}) \quad \text{by} \quad {}_{\mathrm{i}}\#(g):=0\#g=[\mathrm{i}]\#g.$$

Identify $\{1\}\times S^q$ and $\{-1\}\times S^q$ with the *first* and the *second* component of $S^q\sqcup S^q$, respectively. Clearly, for $m\ge 2p+q+3$ the map ${}_{\mathrm{i}}\#$ is a homomorphism.

Remark 2.3 (on comparison to previous work)**.** (a) The Standardization and Group Structure Lemmas 2.1 and 2.2 for $X=D^p_+$ generalize the well-known construction of the connected sum of knots, i.e. of isotopy classes of embeddings $S^q\to S^m$. I could not find either a proof that the connected sum is well-defined for $m=q+2=3$, or reference to such a proof, in [BZ03, CDM, CF63, PS96, Re48, Ro76], for either PL, C^r or C^∞ category (unfortunately,

[9]For a map $f: X\to Y$ and $A\subset X$, $f(A)\subset B\subset Y$, the *abbreviation* $g: A\to B$ of f is defined by $g(x):=f(x)$.

[Ad04, Ka87] and other books are not easily available to me). A non-trivial part of such a proof corresponds to the proof of the Standardization Lemma 2.1.b for $X = D^0_+$. In another formalization the non-trivial part corresponds to proving that 'if long knots are isotopic through knots, then they are isotopic through long knots'.

Similarly, it was not so easy for me to reconstruct omitted proof of [Ha66, Lemma 1.3.b] which is Standardization Lemma 2.1.b for $X = D^0_+$ and $m \ge q + 3$.

Even if this proof is unpublished, it should be known in folklore.

This proof and its generalization from $X = D^0_+$ to $X = D^p_+$ is not hard, see §3.1.

(b) The Standardization and the Group Structure Lemmas 2.1 and 2.2 for $X = S^p$ generalize the well-known construction of the connected sum of links with two *numbered oriented* components, i.e. of isotopy classes of embeddings $S^q \sqcup S^q \to S^m$ [Ha66', Theorem 2.4]. It would be nice to have a published example showing that such a connected sum is not well-defined for $m = q + 2 = 3$, cf. Remark 2.5.d for $p = 0$ and [PS96, Remark before Problem 3.3]. As in (a), a proof that the connected sum is well-defined for $m \ge q + 3$ also seems to be unpublished, cf. [Ha66', 2.5]; a non-trivial part of such a proof corresponds to the proof of the Standardization Lemma 2.1.b for $X = S^0$.

This proof and its generalization from $X = S^0$ to $X = S^p$ is not hard, although more complicated than for $X = D^0_+$, see §3.1.

(c) Also well-known are connected sum group structures on the set $C^{q+2}(S^q)$ of concordance classes of embeddings $S^q \to S^{q+2}$ and on the set $LM^m_{q,q}$ of link homotopy classes of link maps $S^q \sqcup S^q \to S^m$ [Ko88, Proposition 2.3]. (See definition of concordance in §3.1. In knot theory concordance is called 'cobordism', but I use 'concordance' to agree with the rest of topology.)

Remark 2.4 (on the dimension restrictions)**.** (a) *The orbits of the action # consist of different number of elements*, and so *there are no group structures* $(+, 0)$ *on* $E^m(T^{p,q})$ *such that* $f\#g = f + (0\#g)$

- for $p + 1 = q = 2k$ and $m = 2p + q + 2 = 6k$ [Sk08, Classification Theorem and Higher-dimensional Classification Theorem].
- for $q = 3$, $p = 1$ and $m = 2p + q + 2 = 7$ [CS].
- for $q = p = 2$ and $m = 2p + q + 1 = 7$ [CS11, §1].

(b) *There are no group structures on* $E^7(T^{1,3})$ *such that* $r\tau : \pi_3(V_{4,2}) \to E^7(T^{1,3})$ *is a homomorphism. There are no group structures on* $E^7_\#(T^{1,3})$ *such that* $q_\# r\tau : \pi_3(V_{4,2}) \to E^7_\#(T^{1,3})$ *is a homomorphism.* This is so because $r\tau$-preimages of distinct elements consist of different number of elements, and because of the analogous assertion for $q_\# r\tau$ [CS].

(c) *There are no group structures on* $E^m(T^{p,q})$ *such that the Whitney invariant* W *[Sk08, §1], [MAW] is a homomorphism*

- for $p + 1 = q = 2k$ and $m = 2p + q + 2 = 6k$ because W-preimages of distinct elements consist of different number of elements [Sk08, Classification Theorem and Higher-dimensional Classification Theorem]. In this case $W : E^m_\#(T^{p,q}) \to \mathbb{Z}$ is a 1–1 correspondence.
- for $p = q = 2k$ and $m = 2p + q + 1 = 6k + 1$ because by [Bo71, Theorem 5.1] $\operatorname{im} W = \mathbb{Z} \times 0 \cup 0 \times \mathbb{Z}$ which is not a subgroup of the range $\mathbb{Z}^2$ of W. So in this case there are no group structures on $E^m_\#(T^{p,q})$ such that $W : E^m_\#(T^{p,q}) \to \mathbb{Z}^2$ is a homomorphism.

(Note that $\varkappa = 2W$, where $\varkappa$ is defined in [Bo71, §3.9], [CS11, §2.3].)

Remark 2.5 (on the dimension restrictions in the Standardization Lemma 2.1)**.** (a) The analogue of the Standardization Lemma 2.1.a for $X = S^p$, $m = 3$ and $p = q = 1$ holds and follows from the unknottedness of S^2 in S^3 and the Alexander Theorem stating that *any embedding* $T^{1,1} \to S^3$ *extends to an embedding* $S^1 \times D^2 \to S^3$ *or* $D^2 \times S^1 \to S^3$.

(b) If $p \ge q$, $m \ge 2p + q + 3$ and X is either D^p_+ or S^p, then $|E^m(X \times S^q)| = 1$ (by (D) of §1.3 for $X = D^p_+$, and by the Haefliger-Zeeman Unknotting Theorem [Sk08', Theorem 2.8.b]

for $X = S^p$). So the Standardization, the Group Structure and the Triviality Lemmas 2.1.a, 2.2 and 3.1 hold obviously.

(c) If $X = S^p$, then the analogue of the Standardization Lemma 2.1.a

• holds for $m = 2p + q + 2$ by the proof in §3.1; so there is a useful multivalued operation on $E^m(T^{p,q})$, cf. [CS, Definition of a map $\mathbb{Z} \times \mathbb{Z} \times H_1(N) \times E^7(N) \to 2^{E^7(N)}$].

• is false for $m = 2p + q + 1$. (If it were true, then a multivalued operation '+' on $E^m(T^{p,q})$ would be defined by the formula of the Group Structure Lemma 2.2. We would have $W(h) = W(f)+W(g)$ for each $h \in \{f+g\}$, cf. [Sk10', end of §2]. This contradicts to $\operatorname{im} W = \mathbb{Z}\times 0 \cup 0\times\mathbb{Z}$ of Remark 2.4.c for $m = 2p + q + 1$.)

(d) The analogue of the Standardization Lemma 2.1.b for $X = S^p$ and $m = 2p + q + 2$

• holds for embeddings from $\operatorname{im}_{\mathrm{i}}\#$; so there is a group structure on $\operatorname{im}_{\mathrm{i}}\#$, cf. [Av16, Construction of Δ^2 and B^6 in the proof of Lemma 19].

• is false for $p > 0$. (If it were true, then the Group Structure Lemma 2.2 would be true. Indeed, for the deduction in §3.2 of the latter from the Standardization Lemma 2.1 a weaker restriction $m \geq \max\{2p + q + 2, p + q + 3\}$ is sufficient, cf. Remark 3.2.a. So one obtains a contradiction to Remark 2.4.ac for $m = 2p + q + 2$.)

• is conjecturally false for $p = 0$.

2.2 Proof of Theorems 1.2 and 1.7 using Lemmas 2.1, 2.2, 2.6, 2.8

Before reading this subsection a reader might want to get acquainted with the idea by reading the proof of a simpler result in §2.3.

Lemma 2.6 (See proof in §3.4)**.** *For $m \geq p + q + 3$ the following is exact sequence of groups:*

$$\dots \to E^{m+1}(T^{p,q+1}_+) \xrightarrow{\lambda'} \pi_q(S^l) \xrightarrow{\mu'} E^m(T^{p+1,q}_+) \xrightarrow{\nu'} E^m(T^{p,q}_+) \to \dots$$

Here ν' is the restriction-induced map; λ' is defined below and μ' is the composition of τ and the map $\mu'' : \pi_q(S^l) = \pi_q(V_{l+1,1}) \to \pi_q(V_{m-q,p+1})$ induced by 'adding p vectors' inclusion.

Definition of λ'. Take an embedding $f : T^{p,q+1}_+ \to S^{m+1}$. Informally, $\lambda'[f]$ is the obstruction to the existence of a vector field on $f(1_p \times S^{q+1})$ normal to $f(T^{p,q+1}_+)$. The following accurate definition is [Sk15, Definition of Ob in p.9], cf. definition of $\overline{\lambda}$ in §2.4. For sufficiently small $\varepsilon > 0$ take

• a trivialization $t : D^p_+ \times D^{q+1}_- \times D^{l+1} \to S^{m+1}$ of the normal bundle to $f(D^p_+ \times D^{q+1}_-)$ such that $|t(x, y, z) - f(x, y)| = \varepsilon$ for each $(x, y, z) \in D^p_+ \times D^{q+1}_- \times S^l$;

• a unit normal to $f(T^{p,q+1}_+)$ vector field $s : D^{q+1}_+ \to S^{m+1}$ on $f(1_p \times D^{q+1}_+)$.

Let $\lambda'[f]$ be the homotopy class of the map

$$S^q \xrightarrow{\theta} \partial D^{q+1}_+ \xrightarrow{t^{-1}s} D^p_+ \times D^{q+1}_- \times S^l \xrightarrow{\mathrm{pr}_3} S^l, \quad \text{where} \quad \theta(x) := (0, x).$$

This is well-defined because both trivialization t and vector field s are unique up to homotopy.

Definition of the Zeeman homomorphism

$$\zeta = \zeta_{m,n,q} : \pi_q(S^{m-n-1}) \to E^m_U(S^q \sqcup S^n) \quad \text{for} \quad q \leq n.$$

Denote by $\mathrm{i}_{m,q} : S^q \to S^m$ the standard embedding. For a map $x : S^q \to S^{m-n-1}$ representing an element of $\pi_q(S^{m-n-1})$ let

$$\overline{\zeta}_x : S^q \to S^m \quad \text{be the composition} \quad S^q \xrightarrow{x\times \mathrm{i}_{n,q}} S^{m-n-1} \times S^n \xrightarrow{\mathrm{i}} S^m, \quad \text{where} \quad \mathrm{i} := \mathrm{i}_{\,m,m-n-1,n}.$$

We have $\overline{\zeta}_x(S^q) \cap \mathrm{i}_{m,n}(S^n) \subset \mathrm{i}(S^{m-n-1} \times S^n) \cap \mathrm{i}(0_{m-n} \times S^n) = \emptyset$. Let $\zeta[x] := [\overline{\zeta}_x \sqcup \mathrm{i}_{m,n}]$. Clearly, ζ is well-defined, is a homomorphism, and $\lambda\zeta = \mathrm{id}\, \pi_q(S^{m-n-1})$.

Cf. [Sk15, Definition of Ze in p.9]. Note that $\zeta_{m,q,q} = r\tau^m_{0,q} = r\mu'$.

Definition of the homomorphism

$$\sigma = \sigma_{m,p,q} : E^m_0(S^q \sqcup S^{p+q}) \to E^m(T^{p,q}) \quad \text{for} \quad m \geq p+q+3 \quad \text{and} \quad q > 0.$$

Cf. [Sk15, §3, Definition of σ^*]. Denote by $E^m_0(S^q \sqcup S^n) \subset E^m(S^q \sqcup S^n)$ the subset formed by the isotopy classes of embeddings whose restriction to the *first* component is unknotted. Represent an element of $E^m_0(S^q \sqcup S^{p+q})$ by an embedding

$$f : S^q \sqcup S^{p+q} \to S^m \quad \text{such that} \quad f|_{S^q} = \mathrm{i}\,|_{0_{p+1} \times S^q} \quad \text{and} \quad f(S^{p+q}) \cap \mathrm{i}(D^{p+1} \times S^q) = \emptyset.$$

Join $f(S^{p+q})$ to $\mathrm{i}(-1_p \times S^q)$ by an arc whose interior misses $f(S^{p+q}) \cup \mathrm{i}(D^{p+1} \times S^q)$. Let $\sigma[f]$ be the isotopy class of the embedded connected sum of $\mathrm{i}\,|_{S^p \times S^q}$ and $f|_{S^{p+q}}$ along the arc. (The images of these embeddings are not necessarily contained in disjoint balls.)

The map σ is well-defined for $m \geq p+q+3$ and is a homomorphism for $m \geq 2p+q+3$ [Sk15, Lemmas 3.2 and 3.3]. (The main reason for being well-defined is that $E^m_0(S^q \sqcup S^n)$ is in 1–1 correspondence with the set of isotopy classes of embeddings whose restriction to the first component is *standard* [Sk15, Lemma 3.1], cf. [Ha66', Proof of Theorem 7.1].)

Remark 2.7. (a) For $p = 0$ the map σ is not necessarily the identity but is an analogue of the *unframed second Kirby move* [Ma80, §3.1]. For an embedding $f : S^q \sqcup S^q \to S^m$ the image $\sigma[f]$ is the isotopy class of the embedding $T^{0,q} \to S^m$ that

• on $1 \times S^q$ is the 'standard shift' of the first component of f and

• on $-1 \times S^q$ is the embedded connected sum of the components of f (the order of the summands does not play a role).

(b) We have $\sigma(f) + {}_{\mathrm{i}}\#g = \sigma(f\#g)$. See [Sk15, Remark after Claim 3.2] on ${}_{\mathrm{i}}\#$ and σ.

Proof of Theorem 1.2. Consider the following diagram (the dotted arrow $q_\# r$ is required not for this proof but for the proof of Theorem 1.7 below).

$$\begin{tikzcd}
 & \pi_q(S^l) \arrow[r, "\mu'"] \arrow[d, shift right, "\zeta"'] & E^m(T^{p+1,q}_+) \arrow[d, dotted, "q_\# r"] \arrow[dr, "\nu'"] & & \pi_{q-1}(S^l) \arrow[d, shift right, "\zeta"'] \\
E^{m+1}(T^{p,q+1}_+) \arrow[ur, "\lambda'"] \arrow[r, "\zeta\lambda'"'] \arrow[dr, "i\zeta\lambda'"'] & E^m_U(S^q \sqcup S^{p+q}) \arrow[u, dashed, shift right, "\lambda"'] \arrow[r, "q_\#\sigma"'] \arrow[d, "i"] & E^m_\#(T^{p,q}) \arrow[r, "\overline{\nu}"'] & E^m(T^{p,q}_+) \arrow[ur, "\lambda'"] \arrow[r, "\zeta\lambda'"'] \arrow[dr, "i\zeta\lambda'"'] & E^{m-1}_U(S^{q-1} \sqcup S^{p+q-1}) \arrow[u, dashed, shift right, "\lambda"'] \arrow[d, "i"] \\
 & E^m_0(S^q \sqcup S^{p+q}) \arrow[r, "\sigma"] & E^m(T^{p,q}) \arrow[u, "q_\#"] \arrow[ur, "\nu"] & & E^{m-1}_0(S^{q-1} \sqcup S^{p+q-1}) \\
 & E^m(S^{p+q}) \arrow[u, "{}_j\#"] \arrow[ur, "{}_{\mathrm{i}}\#"'] & & &
\end{tikzcd}$$

Here the $\lambda'\mu'\nu'$-sequence is defined in Lemma 2.6, maps ζ and σ are defined above,

• i is the inclusion,

• ν is the restriction-induced map,

• the map $\overline{\nu}$ is well-defined by $\overline{\nu}q_\#(f) := \nu(f)$,

• ${}_j\#g := j\#g$, where the 'standard embedding' $j : S^q \sqcup S^{p+q} \to S^m$ is any embedding whose components are contained in disjoint balls and are isotopic to the inclusions.

The commutativity of the triangles is clear, except for ${}_{\mathrm{i}}\# = \sigma_j\#$, which follows by $\sigma[j] = [\mathrm{i}]$.

The exactness of the $\nu\sigma(i\zeta\lambda')$-sequence is [Sk15, Theorem 1.6].

The map $i \oplus {}_j\#$ is an isomorphism [Ha66', Theorem 2.4]. Hence by the Smoothing Lemma 1.1 and ${}_\mathrm{i}\# = \sigma_j\#$, the first two third-line groups both have $E^m(S^{p+q})$-summands mapped one to the other under σ. Taking quotients by this summands one obtains the exactness of the second line.

Since $\lambda\zeta = \mathrm{id}$, Theorem 1.2 follows by Lemma 2.8 below applied to the first two lines. $\square$

Lemma 2.8 (see proof in §3.5)**.** *Consider the following diagram (of finitely generated abelian groups) whose triangles are commutative:* [10]

$$\begin{tikzcd}
 & B_1 \arrow[r, "b_1"] \arrow[d, "t_1"'] & C_1 \arrow[d, dotted, "r"'] \arrow[dr, "c_1"] & & B_0 \arrow[d, "t_0"] \\
A_1 \arrow[ur, "a_1"] \arrow[r, "a'_1"'] & B'_1 \arrow[u, bend right, dashed, "l_1"'] \arrow[r, "b'_1"'] & C'_1 \arrow[r, "c'_1"'] & A_0 \arrow[ur, "a_0"] \arrow[r, "a'_0"'] & B'_0
\end{tikzcd}.$$

Assume that t_0 is injective, $l_1t_1 = \mathrm{id}\, B_1$ and sequences $a_1b_1c_1a_0$ and $a'_1b'_1c'_1a'_0$ are exact.

Then $b'_1 \ker l_1 \cong \ker l_1$ and groups $C'_1/b'_1 \ker l_1$, C_1 have isomorphic subgroups with isomorphic quotients.

Moreover, if C_1 is finite and $\ker l_1$ is free, then $b'_1 \ker l_1$ is a direct summand in C'_1.

Proof of Theorem 1.7. Part (b) follows by the 'moreover' part of Lemma 2.8 applied to the upper two lines of the diagram from the proof of Theorem 1.2, including the dotted arrow $q_\# r$.

Let us prove part (a). Denote $m = 6l - p$ and $q = 4l - p - 1$. Since $p < l$, we have $m \ge 2p + q + 3$ and $m \ge \frac{2p+4q}{3} + 2$. Hence by (L') of §1.3 we have $K^m_{q,p+q} \cong \mathbb{Z}$ is free. Since $2m = p + 3q + 3 \ge 3q + 4$, by (D) of §1.3 we have $E^m(D^{p+1} \times S^q) \cong \pi_q(V_{m-q,p+1})$. This is finite [CFS, Lemma 1.12]. So (a) follows from (b). $\square$

Remark 2.9. (a) Proofs of Theorem 1.2 and Lemma 2.8 show the following.

The first subgroup of Theorem 1.2 is $q_\#\sigma(K^m_{p,p+q})$. The second subgroups of Theorem 1.2 are $\mathrm{im}(q_\#\sigma\zeta)$ and $\mathrm{im}\,\mu' = \ker \nu'$. The quotients of Theorem 1.2 are $\mathrm{im}\,\nu' = \ker \lambda' = \ker(\zeta\lambda') = \mathrm{im}\,\overline{\nu}$.

In other words, the groups $E^m_\#(T^{p,q})$ and $E^m(T^{p+1,q}_+) \oplus K^m_{p,p+q}$ are associated to the same group for the filtrations

$$0 \subset q_\#\sigma(K^m_{p,p+q}) \subset q_\#\,\mathrm{im}\,\sigma \subset E^m_\#(T^{p,q}) \quad \text{and}$$

$$0 \subset 0 \oplus K^m_{p,p+q} \subset \mathrm{im}\,\mu' \oplus K^m_{p,p+q} \subset E^m(T^{p+1,q}_+) \oplus K^m_{p,p+q}.$$

(b) The square in the diagram from the proof of Theorem 1.2 is not commutative, i.e. $q_\#\sigma\zeta \ne q_\# r\mu'$, for $p = 0$, $q = 4l - 1$ and $m = 6l$, by Theorem 2.13.b because $\zeta = r\mu'$.

(c) I conjecture that the piecewise linear (PL) analogue of Theorem 1.2 holds. This analogue is obtained by replacing $E^m_\#(T^{p,q})$ and $E^m(T^{p+1,q}_+)$ by $E^m_{PL}(T^{p,q})$ and $E^m_{PL}(T^{p+1,q}_+)$; the group $K^m_{q,p+q}$ remains the same.

The PL analogue of the Standardization Lemma 2.1 for $X = S^p$ holds by [Sk07]. The PL analogues of the Standardization Lemma 2.1 for $X = D^p_+$, of the Group Structure and the Triviality Lemmas 2.2 and 3.1 hold with the same proof (there is even a simplification in the proof that $[f] + [f] = 0$). It would be interesting to find the PL analogue of Lemma 1.10.

[10] This version of 5-lemma is simple and possibly known. The existence of the dotted arrow r such that $c_1 = rc'_1$ is not required before the 'moreover' part.

2.3 Appendix: new direct proof of Corollary 1.8.a

In this subsection we present some ideas of proof of Theorem 1.2 by presenting a direct proof of Corollary 1.8.a. (This proof is simpler than that sketched in §1.2 for $p = 1$, but more complicated than the classical proof for $p = 0$.) Formally, this subsection is not used later except that §2.4 uses statement of Lemma 2.10.a.

Definition of the following diagram for $m \ge p + q + 3$**.**

$$\begin{tikzcd}
\pi_{q+1}(V_{m-q,p}) \arrow[r, "\lambda''"] \arrow[d, "\tau_p"] & \pi_q(S^l) \arrow[r, "\mu''"] \arrow[dr, "\sigma'"'] & \pi_q(V_{m-q,p+1}) \arrow[r, "\nu''"] \arrow[d, "q_\# r\tau_{p+1}"] & \pi_q(V_{m-q,p}) \arrow[r, "\lambda''"] \arrow[d, "\tau_p"] & \pi_{q-1}(S^l) . \\
E^{m+1}(T^{p,q+1}_+) \arrow[ur, "\lambda'"'] & & E^m_\#(T^{p,q}) \arrow[r, "\overline{\nu}"'] & E^m(T^{p,q}_+) \arrow[ur, "\lambda'"'] &
\end{tikzcd}$$

Here $q_\#, r, \lambda'$ and $\tau_p := \tau^m_{p,q}$ are defined in §§1.1, 1.2, 2.2,

• the $\mu''\nu''\lambda''$ sequence is the exact sequence of the 'forgetting the last vector' bundle $S^l \to V_{m-q,p+1} \to V_{m-q,p}$,

• the map $\overline{\nu}$ is well-defined by $\overline{\nu} q_\# \operatorname{pr}[f] := [f|_{T^{p,q}_+}]$.

• the map $\sigma' = \sigma'_p$ is defined as follows. For a map $x : S^q \to S^l$ representing an element of $\pi_q(S^l)$ let ζ_x be the composition

$$D^{p+1} \times S^q \overset{\mathrm{i}_{p+q,p,q}\times x}{\to} D^{p+q+1} \times S^l \overset{\mathrm{i}}{\to} S^m \overset{h}{\to} S^m,$$

where $\mathrm{i} := \mathrm{i}_{\,m,p+q,l}$ and h is the exchange of the first $l+1$ and the last $p+q+1$ coordinates. The image of ζ_x is contained in $h\,\mathrm{i}(D^{p+q+1} \times S^l)$ and so is disjoint from $S^{p+q} = h\,\mathrm{i}(S^{p+q} \times 0_{l+1})$. Join S^{p+q} to $\zeta_x(-1_p \times S^q)$ by an arc whose interior misses $S^{p+q} \cup \zeta_x(D^{p+1} \times S^q)$. Let $\sigma'[x]$ be the equivalence class of the embedded connected sum of the inclusion $S^{p+q} \subset S^m$ and $\zeta_x|_{T^{p,q}}$ along the arc. Clearly, σ' is well-defined for $m \ge p + q + 3$, is a homomorphism (and $\sigma' = q_\#\sigma(h^{-1}\zeta)$ in the notation of §2.2).

Lemma 2.10. *(a) The $\sigma'\overline{\nu}\lambda'$-sequence is exact for $2m \ge p + 3q + 4$.*

(b) The right τ_p is an isomorphism for $2m \ge 3q + 5$ and an epimorphism $2m \ge 3q + 4$.

(c) The diagram is commutative up to sign for $m \ge 2p + q + 2$ and $2m \ge 2p + 3q + 4$.

Corollary 1.8.a for $p \ge 1$ follows from Lemma 2.10 and 5-lemma. Corollary 1.8.a for $p = 0$ follows from Lemma 2.10 because $V_{m-q,0}$ is a point, so $q_\# r\tau_1$ is the composition of the isomorphisms σ' and $(\mu'')^{-1}$.

Lemma 2.10.a follows from (L) of §1.3 and [Sk15, Theorem 1.6] (restated in the proof of Theorem 1.2 in §2.2). In §2.4 we present a simpler direct proof of Lemma 2.10.a for $m \ge 2p+q+3$ and $2m \ge 2p+3q+4$ recovered from [Sk06] (this weaker result is sufficient for Corollary 1.8.a). This illustrates ideas of proof of the deeper result [Sk15, Theorem 1.6] whose full strength is only used in §2.2.

Lemma 2.10.b follows by Theorem 1.10 and (D) of §1.3. [11]

An embedding $f : S^n \times X \to S^m$ is *reflection-symmetric* if $f \circ (R_1 \times \operatorname{id} X) = R_1 \circ f$.

Proof of Lemma 2.10.c. Clearly, $\tau_p\nu'' = \overline{\nu} q_\# r\tau_{p+1}$. By definitions of λ' (§2.2) and of λ'' (§3.4) $\lambda'' = \tau_p\lambda'$. So it remains to prove that $\sigma'_p = q_\# r\tau_{p+1}\mu''_p$. Take any $x \in \pi_q(S^l)$.

First we prove the case $p = 0$ for $2m \ge 3q+4$. An embedding $f' : T^{0,q} \to S^m$ representing $\sigma'_0 x$ is obtained from an embedding $f : T^{0,q} \to S^m$ representing $r\tau_1\mu''_0$ by the *unframed second Kirby*

[11]The required case of Theorem 1.10 is trivial, so the assertion follows just by (D) of §1.3, because $E^{m+1}(S^{q+1}) = 0$ implies that every isotopy $S^q \times I \to S^m \times I$ between standard embeddings is isotopic to the identical one.

move defined in Remark 2.7.a. Denote by $\lambda_- f$ the linking coefficient, i.e. the homotopy class of the *second* component in the complement to the first component. Define $\lambda_- f'$ analogously. Clearly, $\lambda_- f' = \lambda_- f$. The restrictions of f and f' to the first component are isotopic to the standard embedding. Since $2m \ge 3q+4$, by (L) of §1.3 $q_\#[f'] = q_\#[f]$.

A representative of $r\tau_{p+1}\mu''_p x$ is a reflection-symmetric extension of a representative of $r\tau_p\mu''_{p-1}x$. A representative of $\sigma'_p x$ is a reflection-symmetric extension of a representative of $\sigma'_{p-1}x$. Since $m \ge 2p+q+2$, '$\sigma'_p = q_\# r\tau_{p+1}\mu''_p$ for $2m \ge 2p+3q+4$' follows from '$\sigma'_0 = q_\# r\tau_1\mu''_0$ for $2m \ge 3q+4$' by the following Extension Lemma 2.11. $\square$

Lemma 2.11 (Extension Lemma)**.** *If $m \ge 2p+q+2$ and $f_p, g_p : T^{p,q} \to S^m$ are reflection-symmetric extensions of concordant embeddings $f, g : T^{p-1,q} \to S^{m-1}$, then f_p and g_p are concordant.*

Proof. Let F be a concordance between f and g. Identify

$$D^m_+ \bigcup_{S^{m-1}=S^{m-1}\times 0} S^{m-1}\times I \bigcup_{S^{m-1}\times 1=\widetilde{S^{m-1}}} \widetilde{D^m_+} \quad \text{with} \quad S^m \quad \text{and}$$

$$T^{p,q}_+ \bigcup_{T^{p-1,q}=T^{p-1,q}\times 0} T^{p-1,q}\times I \bigcup_{T^{p-1,q}\times 1=\widetilde{T^{p-1,q}}} \widetilde{T^{p,q}_+} \quad \text{with} \quad T^{p,q},$$

where by $\widetilde{A}$ we denote a copy of A. Let $F' : T^{p,q} \to S^m$ be an embedding obtained from $f_p|_{T^{p,q}_+} \cup F \cup g_p|_{T^{p,q}_+}$ by these identifications. Define an embedding

$$i_\varepsilon : D^q \to S^{q+1} \quad \text{by} \quad i_\varepsilon(x) := (\varepsilon x, \sqrt{1-\varepsilon^2|x|^2}).$$

Since $m \ge 2p+q+2$, analogously to the Standardization Lemma 2.1.a there are $\varepsilon > 0$ and an embedding $\psi : D^{p+1}\times D^q \to S^m$ such that $\psi|_{S^p\times D^q} = F' \circ (\mathrm{id}\, S^p \times i_\varepsilon)$. Let

$$\Sigma := (T^{p,q} - S^p \times i_\varepsilon(\mathrm{Int}\, D^q)) \bigcup_{S^p\times i_\varepsilon(S^{q-1})=\widetilde{S^p\times S^{q-1}}} \widetilde{D^{p+1}\times S^{q-1}}.$$

Clearly, $\Sigma \cong S^{p+q}$. Let $\gamma : \mathrm{Con}\,\Sigma \to S^m \times I$ be the cone over $(F'\cup\psi)|_\Sigma$. Identify

$$T^{p+1,q} \quad \text{with} \quad D^{p+1}\times D^q \bigcup_{D^{p+1}\times S^{q-1}=\widetilde{D^{p+1}\times S^{q-1}}\times 0} \mathrm{Con}\,\Sigma.$$

Take a piecewise smooth embedding $T^{p+1,q}_+ \to S^m \times I$ obtained from $F'\cup\psi\cup\gamma$ by this identification. This embedding can clearly be shifted to a proper piecewise smooth concordance F_+ between $f_p|_{T^{p,q}_+}$ and $g_p|_{T^{p,q}_+}$, smooth outside a ball.

The complete obstruction to smoothing F_+ is in $E^m(S^{p+q})$ [BH70, Bo71]. If we change concordance F by connected sum with an embedding $h : S^{p+q} \to S^{m-1}\times I$, then F_+ changes by a connected sum with the cone over h. Hence the obstruction to smoothing F_+ changes by adding $[h] \in E^m(S^{p+q})$ [BH70, Bo71]. Therefore by changing F modulo the ends we can make F_+ a smooth concordance (in particular, orthogonal to the boundary). So we may assume that F_+ is a smooth concordance.

Define F_- by symmetry to F_+. The two proper concordances F_+ and F_- fit together to give the required concordance between f_p and g_p. $\square$

2.4 Appendix: new direct proof of Lemma 2.10.a

Abbreviate $\mathrm{i} = \mathrm{i}_{m,p,q}$. Take the 'standard' homotopy equivalence $h : S^m - \mathrm{i}(1_p \times S^q) \to S^{m-q-1}$.

Definition of $\lambda(f) \in \pi_{p+q}(S^{m-q-1})$ for an embedding $f : T^{p,q} \to S^m$ coinciding with i *on* $T^{p,q}_+$. The restrictions of f and i onto $T^{p,q}_-$ coincide on the boundary and so form a map

$$\widehat{f} : T^{p,q} \to S^m - \mathrm{i}(1_p \times S^q) \xrightarrow{h} S^{m-q-1}.$$

Clearly, $\widehat{f}|_{\mathrm{i}(-1_p \times S^q)}$ is null-homotopic. Since $m \geq 2p + q + 3$, by general position the map

$$T^{p,q} \to \frac{T^{p,q}}{-1_p \times S^q \vee S^p \times 1_q} \cong S^{p+q}$$

induces a 1–1 correspondence between $\pi_{p+q}(S^{m-q-1})$ and homotopy classes of maps $T^{p,q} \to S^{m-q-1}$ null-homotopic on $-1_p \times S^q$. [12] Let $\lambda(f)$ be the homotopy class corresponding to the homotopy class of $-1_p \times S^q$. [13]

Definition of $\overline{\lambda}(f) \in \pi_{p+q-1}(S^{m-q-1})$ for an embedding $f : T^{p,q}_+ \to S^m$ coinciding with i *on* $1_p \times S^q$. Since $m \geq 2p + q + 2$, by general position the map

$$\partial T^{p,q}_+ \overset{\cong}{\to} T^{p,q-1} \to T^{p,q-1}/S^p \times 1_q \overset{\cong}{\to} S^{p+q-1} \vee S^{q-1}$$

induces a 1–1 correspondence between homotopy classes of maps $\partial T^{p,q}_+ \to S^{m-q-1}$ and $S^{p+q-1} \vee S^{q-1} \to S^{m-q-1}$. Let $\overline{\lambda}(f)$ be the image under the restriction map of the homotopy class corresponding to the homotopy class of the map

$$f|_{\partial T^{p,q}_+} : \partial T^{p,q}_+ \to S^m - \mathrm{i}(1_p \times S^q) \xrightarrow{h} S^{m-q-1}.$$

Lemma 2.12. *Assume that $m \geq 2p + q + 3$ and $2m \geq 2p + 3q + 4$.*

(a) If embeddings $f, g : T^{p,q} \to S^m$ coincide with i *on $T^{p,q}_+$ and $\lambda(f) = \lambda(g)$, then $q_\#[f] = q_\#[g]$.*

(b) For each $y \in \pi_q(S^l)$ there is an embedding $g : T^{p,q} \to S^m$ representing $\sigma'(y)$, coinciding with i *on $T^{p,q}_+$ and such that $\lambda(g) = \pm\Sigma^p y$.*

(c) For each embedding $f : T^{p,q}_+ \to S^m$ coinciding with i *on $1_p \times S^q$ we have $\overline{\lambda}(f) = \pm\Sigma^p \lambda'(f)$. (Recall that λ' is defined in §2.2.)*

Proof of (a). Since $\lambda(f) = \lambda(g)$, the abbreviations $f, g : T^{p,q}_- \to S^m - \mathrm{i}(1_p \times S^q)$ are homotopic relative to the boundary. Since $q - 1 \geq 2(p+q) - m + 1$ and $m - q - 2 \geq 2(p+q) - m + 2$, these abbreviations are PL isotopic by [Ir65]. Since $2m \geq 3q + 4$, there is a unique obstruction to smoothing such a PL isotopy, assuming values in $E^m(S^{p+q})$. This obstruction can be killed by embedded connected sum of f with an embedding $S^{p+q} \to S^m$. Hence $q_\#[f] = q_\#[g]$. $\square$

Proof of (b). Take a map $y : S^q \to S^l$ representing y. Take the linear homotopy between the map $S^l \to 0_{l+1} \in \mathbb{R}^{l+1}$ and the composition of y with the inclusion $S^l \subset \mathbb{R}^{l+1}$. This

[12] Looking at the Puppe sequence one can see that $m \geq 2p + q + 2$ is sufficient.

[13] In this definition of $\lambda(f)$ one can avoid using the triviality of the restriction to $\mathrm{i}(-1_p \times S^q)$, analogously to the definition of $\overline{\lambda}(f)$ given below. However, the above definition is more convenient for Lemma 2.12.
Alternative definition of $\lambda(f)$. Denote $B^{p+q} := T^{p,q} - (\operatorname{Int} D^p_+ \times S^q \cup S^p \times \operatorname{Int} D^q_+)$. Since $m \geq 2p + q + 2$, by general position making an isotopy we may assume that $f = \mathrm{i}$ on $T^{p,q} - \operatorname{Int} B^{p+q}$. Hence $f|_{B^{p+q}}$ and $\mathrm{i}|_{B^{p+q}}$ form a map $S^{p+q} \to S^m - \mathrm{i}(1_p \times S^q) \xrightarrow{h} S^{m-q-1}$. Let $\lambda(f)$ be the homotopy class of this map. Since $m \geq 2p + q + 3$, this is well-defined, i.e., is independent of the isotopy used in the definition.
There is analogous *alternative definition of $\overline{\lambda}(f)$.*
The element $\lambda(f)$ is not an isotopy invariant of f, as opposed to $\overline{\lambda}(f)$.

homotopy defines an isotopy between $\mathfrak{i}$ and the embedding ζ_y from the definition of σ'. This isotopy is 'covered' by an isotopy of S^m. The latter isotopy carries the representative of $\sigma'(y)$ constructed in the definition to an embedding $T^{p,q} \to S^m$ which we denote by g. Clearly, $g = \mathfrak{i}$ on $T^{p,q}_+$. Clearly, $\lambda(g)$ equals to the homotopy class of 'the image' of S^{p+q} under this isotopy in $S^m - \mathfrak{i}(1_p \times S^q) \sim S^{m-q-1}$. This equals to the homotopy class of S^{p+q} in $S^m - \zeta_y(1_p \times S^q) \sim S^{m-q-1}$. Since $q \le 2l-2$, by the Freudenthal Suspension Theorem the group $\pi_q(S^l)$ is stable. Hence by [Ke59, Lemma 5.1] the latter homotopy class equals to the $\pm\Sigma^p$-image of the homotopy class of $\zeta_y(1_p \times S^q)$ in $S^m - S^{p+q} \sim S^l$. The latter class equals to y, so $\lambda(g) = \pm\Sigma^p y$. $\square$

Proof of (c). (This is a simpler 'algebraic' analogue of [Sk15, Lemma 3.8].)

Take an embedding $f : T^{p,q}_+ \to S^m$. Since $2m \ge 3q+4$, making an isotopy of S^m we may assume that $f = \mathfrak{i}$ on $1_p \times S^q$. Clearly, $\overline{\lambda}(f)$ is the homotopy class of the composition

$$S^{p+q-1} \overset{\cong}{\to} \partial D^p_+ \times D^q_- \cup D^p_- \times \partial D^q_- \overset{f\cup\mathfrak{i}}{\to} S^m - \mathfrak{i}(1_p \times S^q) \overset{h}{\to} S^{m-q-1},$$

where the map $f \cup \mathfrak{i}$ is formed by the abbreviations of f and $\mathfrak{i}$ which agree on the boundary.

Identify in the natural way

- S^{p+q-1} with $\partial D^p_+ \times D^q_- \cup D^p_- \times \partial D^q_-$;
- $D^p_+ \times D^q_-$ with $D^q_- \times D^p_+$; and
- S^{m-q-1} with $\partial(D^p_+ \times D^{l+1}) = \partial D^p_+ * S^l$, where $X * Y = \dfrac{X \times I \times Y}{X \times 0 \times y, x \times 1 \times Y}$.

Use the notation t, s from definition of λ' in §2.2. Since $f = \mathfrak{i}$ on $1_p \times S^q$, we may assume that $ht = \mathrm{pr}_2$ on $D^q_+ \times \partial(D^p_+ \times D^{l+1})$. So the above composition representing $\overline{\lambda}(f)$ maps

- $\partial D^p_+ \times D^q_- = f^{-1}t(D^q_- \times \partial D^p_+ \times 0_{l+1})$ to $\partial D^p_+ \times 0 \times [S^l] \subset \partial D^p_+ * S^l$ as $\mathrm{pr}_2\, t^{-1} f$;
- $-1_p \times \partial D^q_-$ to $[\partial D^p_+] \times 1 \times S^l \subset \partial D^p_+ * S^l$ as $\mathrm{pr}_3\, t^{-1} s\, \mathrm{pr}_2$;
- $D^p_- \times \partial D^q_-$ 'linearly' to the join $\partial D^p_+ * S^l$.

Therefore $\pm\overline{\lambda}(f) = \Sigma^p[\mathrm{pr}_3\, t^{-1} s\theta] = \Sigma^p \lambda'(f)$. $\square$

In the rest of this subsection we prove Lemma 2.10.a for $m \ge 2p+q+3$ and $2m \ge 2p+3q+4$

The exactness at $E^m_\#(T^{p,q})$. Clearly, $\overline{\nu}\sigma' = 0$.

Let $f : T^{p,q} \to S^m$ be an embedding such that $\overline{\nu} q_\#[f] = 0$. Making an isotopy of S^m we may assume that $f = \mathfrak{i}$ on $T^{p,q}_+$. Since $q \le 2l-1$, by the Freudenthal Suspension Theorem there is $x \in \pi_q(S^l)$ such that $\Sigma^p x = \lambda(f)$. So by Lemma 2.12.b there is an embedding $g : T^{p,q} \to S^m$ representing $\pm\sigma'(x)$, coinciding with $\mathfrak{i}$ on $T^{p,q}_+$ and such that $\lambda(g) = \Sigma^p x$. Then by Lemma 2.12.a $q_\#[f] = q_\#[g] = \sigma'(\pm x)$. $\square$

The exactness at $E^m(T^{p,q}_+)$. Take any $z \in \ker \lambda'$. Since $2m \ge 3q+4$, by Lemma 2.10 the right τ_p of the diagram from the beginning of §2.3 is an epimorphism. Hence there is $z_1 \in \pi_q(V_{m-q,p})$ such that $z = \tau_p z_1$. By Lemma 2.10.c $\lambda'' z_1 = \lambda' z = 0$. Then by exactness there is $z_2 \in \pi_q(V_{m-q,p+1})$ such that $z_1 = \nu'' z_2$. Then by Lemma 2.10.c $z = \tau_p z_1 = \tau_p \nu'' z_2 = \overline{\nu}(q_\# r \tau_{p+1} z_2)$.

Take an embedding $f : T^{p,q} \to S^m$. Then $f|_{T^{p,q}_-}$ gives a null-homotopy of $\overline{\lambda}(f)$. Since $q \le 2l-2$, by the Freudenthal Suspension Theorem the homomorphism Σ^p of Lemma 2.12.c is injective. So $\lambda'[f] = 0$. $\square$

Proof that $\ker \sigma' \subset \operatorname{im} \lambda'$. Take any $z \in \ker \sigma'$. Take a representative g of $\sigma'(x) = 0$ given by Lemma 2.12.b. Represent $S^{a+1} = D^{a+1}_0 \cup S^a \times I \cup D^{a+1}_1$. The restriction $T^{p,q}_+ \times I \to S^m \times I$ of an isotopy between g and the standard embedding is an isotopy between standard embeddings. So this restriction can be 'capped' to give an embedding $G : T^{p,q+1}_+ \to S^{m+1}$. Since $2m \ge 3q+5$, we may assume that $G = \mathfrak{i}$ on $1_p \times S^q$. We have

$$\Sigma^p \lambda'(G) \overset{(1)}{=} \pm\overline{\lambda}(G) \overset{(2)}{=} \pm\lambda(g) \overset{(3)}{=} \Sigma^p x.$$

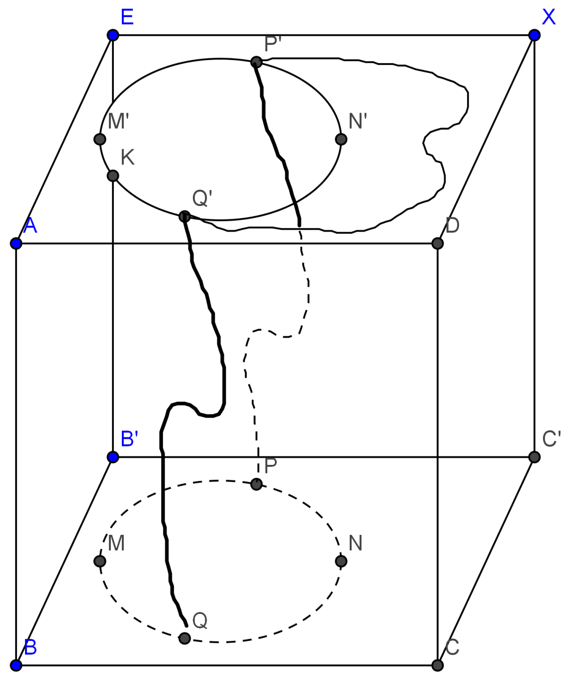

Figure 1: *To the proofs that* $\ker\sigma' \subset \operatorname{im}\lambda'$ *and* $\sigma'\lambda' = 0$. The cube stands for $S^m \times I$; its horizontal faces stand for $S^m \times k$, $k = 0, 1$. The ellipses stand for $\mathrm{i}(T^{p,q}) \times k$, $k = 0, 1$. The curved line $P'Q'$ stands for $g(D^p_- \times S^q) \times 1$. The curved lines PP' and QQ' stand for the image of $\partial D^p_- \times S^q \times I$ under the isotopy between standard embeddings. The points M, N, M', N' stand for $\mathrm{i}(\pm 1_p \times S^q) \times k$, $k = 0, 1$.

Here equalities (1) and (3) follow by Lemma 2.12.bc and equality (2) is proved below. Since $q \le 2l - 2$, by the Freudenthal Suspension Theorem Σ^p injective. So $x = \pm\lambda'(G)$.

Let us prove equality (2). Denote $\Delta^{q+1} := \mathrm{i}(1_p \times D^{q+1})$. Then $\Delta^{q+1} \cap \mathrm{i}(T^{p,q}_-) = \emptyset$ and we may assume that g is transverse to Δ^{q+1}. Since $q \le 2l - 2$, by the Freudenthal Suspension Theorem the group $\pi_q(S^l)$ is stable. So $\lambda(g)$ corresponds under Pontryagin construction and (de)suspension to the framed intersection $\Delta^{q+1} \cap g(T^{p,q}_-) \subset \Delta^{q+1}$, for certain framings on Δ^{q+1} and on $g(T^{p,q}_-)$. Analogously, we may assume that G is transverse to $\Delta^{q+1} \times I \subset S^m \times I \subset S^{m+1}$. So $\overline{\lambda}(G)$ corresponds under Pontryagin construction and (de)suspension to the framed intersection $\Delta^{q+1} \times I \cap G(\partial T^{p,q}_- \times I) \subset \Delta^{q+1} \times I$, for certain framings on $\Delta^{q+1} \times I$ and on $G(\partial T^{p,q}_- \times I)$. Consider the framed intersection $\Delta^{q+1} \times I \cap G(T^{p,q}_- \times I) \subset \Delta^{q+1} \times I$, for certain framings on $\Delta^{q+1} \times I$ and on $G(T^{p,q}_- \times I)$ of which the above framings are the restrictions. The latter framed intersection is a framed cobordism between the first two. Thus $\overline{\lambda}(G) = \lambda(g)$. $\square$

Proof that $\sigma'\lambda' = 0$. [14] (This proof suggests an alternative definition of $\zeta\lambda'$ allowing minor simplification in [Sk15].) Take an embedding $F : T^{p,q+1}_+ \to S^{m+1}$. Represent $S^{a+1} = D^{a+1}_0 \cup S^a \times I \cup D^{a+1}_1$. Analogously to the Standardization Lemma 2.1 making an isotopy we can assume that

- $F|_{D^p_+ \times S^q \times I}$ is a concordance between standard embeddings,
- $F|_{D^p_+ \times D^{q+1}_k}$ is the standard embedding into D^{m+1}_k for each $k = 0, 1$.

Since $q \le 2l - 1$, by the Freudenthal Suspension Theorem there is $x \in \pi_q(S^l)$ such that $\Sigma^p x = \overline{\lambda}(F)$. By Lemma 2.12.b there is an embedding $g : T^{p,q} \to S^m$ representing $\pm\sigma'(x)$, coinciding with i on $T^{p,q}_+$ and such that $\lambda(g) = \Sigma^p x$.

[14] If $2m \ge 3q + 5$, by Lemma 2.10.b the left τ_p is an epimorphism. Then for $2m \ge 2p + 3q + 4$ by Lemma 2.10.c $\sigma'\lambda' = 0$.

The concordance $F|_{D^p_+\times S^q\times I}$ is ambient, so there is a diffeomorphism

$$\Phi : S^m \times I \to S^m \times I \quad \text{such that} \quad \Phi(F(y,0),t) = F(y,t) \quad \text{for each } y \in D^p_+ \times S^q,\ t \in I.$$

So the standard embedding i is concordant to an embedding $f : T^{p,q} \to S^m \times 1 = S^m$ defined by $f(y) := \Phi(F_0(y),1)$. The class $\lambda(f)$ corresponds to the homotopy class of

$$\mathrm{i}\,|_{T^{p,q}_-} \cup f|_{T^{p,q}_-} : T^{p,q} \to S^m \times 1 - \mathrm{i}(\operatorname{Int} T^{p,q}_+) \times 1 \sim S^{m-q-1}.$$

We have $\lambda(f) = \overline{\lambda}(F) = \Sigma^p x = \lambda(g)$, where the first equality is analogous to the equality (2) in the previous proof. Besides, f and g are standard on $T^{p,q}_+$. Therefore by Lemma 2.12.a $\sigma'\lambda'(F) = q_\#[g] = q_\#[f] = 0$. $\square$

2.5 An isomorphism of Conjecture 1.3 for $p = 0$

Let $\iota_n \in \pi_n(S^n)$ be the standard generator, $k > 0$ any integer and $w := [\iota_{2k}, \iota_{2k}] \in \pi_{4k-1}(S^{2k})$.

Theorem 2.13. *Assume that $q = 4k-1$ and $m = (3q+3)/2 = 6k$.*

(a) The following map is an isomorphism:

$$r \oplus \sigma|_{K^m_{q,q}} \oplus \mathrm{i}\# : E^m(D^1 \times S^q) \oplus K^m_{q,q} \oplus E^m(S^q) \to E^m(T^{0,q}).$$

(b) $q_\#\sigma\zeta w \neq q_\#\zeta w \in E^m_\#(T^{0,q})$.

Consider maps

$$\begin{tikzcd} E^m(S^q) \arrow[r, "s_\pm"'] & E^m(T^{0,q}) \arrow[l, bend right, "r_\pm"'] \arrow[r, "\lambda_\pm"'] & \pi_q(S^{m-q-1}) \arrow[l, bend right, "\zeta"'] \arrow[r, "H"] & \pi_q(S^{2(m-q)-3}) \end{tikzcd}, \quad \text{where}$$

• $r_\pm$ is 'the knotting class of the component', i.e. $r_\pm$ is induced by the inclusion $S^q = \pm 1 \times S^q \subset T^{0,q}$,

• $s_\pm$ is the embedded connected summation with the $\pm$ component, so that $s_- = \mathrm{i}\#$,

• $\lambda_\pm$ is the linking coefficient, i.e. the homotopy class of the $\pm$-component in the complement to the other component (see accurate definition in [MAL], [Sk08', §3]),

• H is the generalized Hopf invariant (here it is only used only for $2m = 3q+3$ where it is the ordinary Hopf invariant assuming values in $\pi_q(S^q) \cong \mathbb{Z}$).

Identify $E^{6k}(S^{4k-1})$ and $\pi_q(S^q)$ with $\mathbb{Z}$ by the isomorphism of [Ha66, Sk08] and by the isomorphism carrying ι_q to $+1$.

Recall that $Hw = 2$ [Po85, Lecture 6, (7)].

Lemma 2.14 (Symmetry Lemma)**.** *(a) $\lambda_-\zeta = ((-1)^{m-q}\iota_{m-q-1}) \circ \lambda_+\zeta$.*

(b) If $q = 4k-1$ and $m = (3q+3)/2 = 6k$, then $H\lambda_-\zeta = H\lambda_+\zeta$. [15]

Proof. Part (a) follows because for a map $x : S^q \to S^{m-q-1}$ the link obtained from $\overline{\zeta}_x$ by exchange of components is isotopic to $\overline{\zeta}_{Sx}$, where S is the symmetry w.r.t the origin.

Part (b) follows because

$$H\lambda_-\zeta \overset{(1)}{=} H[(-\iota_{2k}) \circ \lambda_+\zeta] \overset{(2)}{=} H[-\lambda_+\zeta + wH\lambda_+\zeta] \overset{(3)}{=} H\lambda_+\zeta.$$

Here (1) is (a), (2) is [Po85, Complement to Lecture 6, (10)] and (3) follows by $Hw = 2$. $\square$

[15] This and $\Sigma\lambda_- = \Sigma\lambda_+$ imply that $\lambda_-\zeta = \lambda_+\zeta$, which we do not need.

Proof of Theorem 2.13.b assuming the Calculation Lemma 2.15. The result follows because $\lambda_+ s_- = \lambda_+$ and

$$H\lambda_+\zeta w \overset{(1)}{=} Hw = 2 \neq 6 = 3Hw \overset{(4)}{=} (H\lambda_+ + 2H\lambda_-)\zeta w \overset{(5)}{=} H\lambda_+\sigma\zeta w, \quad \text{where}$$

- (1) follows because $\lambda_+\zeta = \mathrm{id}\,\pi_q(S^{m-q-1})$;
- (4) follows because $\lambda_+\zeta = \mathrm{id}\,\pi_q(S^{m-q-1})$ and by the Symmetry Lemma 2.14.b;
- (5) is implied by the following Calculation Lemma 2.15. $\square$

Lemma 2.15 (Calculation Lemma; proof is postponed)**.** *If $q = 4k-1$ and $m = (3q+3)/2 = 6k$, then*

$$\lambda_-\sigma = \lambda_-, \quad r_-\sigma = r_- + \frac{H\lambda_+ + H\lambda_-}{2} \quad \text{and} \quad H\lambda_+\sigma = H\lambda_+ + 2H\lambda_-.$$

Proof of Theorem 2.13.a assuming the Calculation Lemma 2.15. Consider the diagram

$$\begin{tikzcd}
E \oplus E^m(T^{1,q}_+) \oplus K \arrow[r, "s_-\oplus r\oplus\sigma|_K"] \arrow[d, "\varphi:=\mathrm{id}\,E\oplus[r_+r\oplus\lambda_+r]\oplus\mathrm{id}\,K"'] & E^m(T^{0,q}) \arrow[r, "r_-\oplus r_+\oplus\Pi"] & E \oplus E \oplus E^m_U(T^{0,q}) \arrow[d, "\mathrm{id}\,E\oplus\mathrm{id}\,E\oplus[\lambda_+\oplus\varkappa]"] \\
E \oplus [E \oplus \pi] \oplus K \arrow[rr, "\mathrm{id}\,E\oplus\mathrm{id}\,E\oplus\begin{pmatrix}\mathrm{id}\,\pi & 0 \\ \lambda_+\sigma & -\mathrm{id}\,K\end{pmatrix}"'] & & E \oplus E \oplus [\pi \oplus K]
\end{tikzcd} \quad , \quad \text{where}$$

$$K = K^m_{q,q}, \quad E = E^m(S^q), \quad \pi = \pi_q(S^{m-q-1}), \quad \Pi x = x - s_+r_+x - s_-r_-x \quad \text{and} \quad \varkappa x = x - \zeta\lambda_+ x.$$

Clearly, all the horizontal arrows except $s_- \oplus r \oplus \sigma|_K$ are isomorphisms. The vertical arrows are isomorphisms, see Remark 1.11.a, case $p = 0$. Thus it suffices to prove that the diagram is commutative.

Clearly, the diagram is commutative 'over $\varphi^{-1}(E \oplus [E \oplus 0] \oplus 0)$'. The commutativity 'over $\varphi^{-1}(0 \oplus [0 \oplus \pi] \oplus 0)$' holds because $\lambda_+ s_\pm = 0$, so $\lambda_+\Pi(s_-g + rf' + \sigma f) = \lambda_+ rf' + \lambda_+\sigma f$. The commutativity 'over $\varphi^{-1}(0 \oplus 0 \oplus K)$' holds because

$$\Pi s_- = 0, \quad \varkappa\Pi r = 0 \quad \text{and} \quad \varkappa\Pi\sigma|_K = -\,\mathrm{id}\,K, \quad \text{so} \quad \varkappa\Pi(s_-g + rf' + \sigma f) = -f.$$

Below we prove the first three equalities of the above.

The equality $\Pi s_- = 0$ *holds* because $r_-s_- = \mathrm{id}\,E$.

The equality $\varkappa\Pi r = 0$ *holds* because for each $x \in E^m(T^{1,q}_+)$

$$\varkappa\Pi r x \overset{(1)}{=} \varkappa\Pi r\tau^m_{0,q} y \overset{(2)}{=} \varkappa\Pi\zeta y \overset{(3)}{=} \varkappa\zeta y \overset{(4)}{=} 0, \quad \text{where}$$

- equality (1) holds for some $y \in \pi$ by Theorem 1.10 because $\lambda_+ r\tau^m_{0,q} = \mathrm{id}\,\pi$,
- equality (2) holds because $r\tau^m_{0,q} = \zeta$,
- equality (3) holds because $r_+\zeta = r_-\zeta = 0$, and
- equality (4) holds because $\lambda_+\zeta = \mathrm{id}\,\pi$.

Proof that $\varkappa\Pi\sigma|_K = -\,\mathrm{id}\,K$. Take any $x \in K = \ker(\lambda_+ \oplus r_+ \oplus r_-)$. Then $\Pi x = x$. Also, $\Pi\zeta = \zeta$. So it suffices to prove that

$$\widehat{x} := x + \Pi\sigma x - \zeta\lambda_+\sigma x = 0.$$

Recall that *embeddings $T^{0,4k-1} \to \mathbb{R}^{6k}$ are isotopic if and only if their $r_\pm$- and $\lambda_\pm$-invariants coincide* [Ha62', §3]. The map $H\lambda_- : K \to \mathbb{Z}$ is an isomorphism. So we are only required to prove that $r_\pm\widehat{x} = 0$, $\lambda_+\widehat{x} = 0$ and $H\lambda_-\widehat{x} = 0$.

Since $r_\pm x = 0$, $r_\pm\Pi = 0$ and $r_\pm\zeta = 0$, we have $r_\pm\widehat{x} = 0$.

Since $\lambda_+ x = 0$, $\lambda_+\Pi = \lambda_+$ and $\lambda_+\zeta = \mathrm{id}\,\pi$, we have $\lambda_+\widehat{x} = 0$.

By the Calculation Lemma 2.15 $H\lambda_-\sigma x = H\lambda_- x$ and $H\lambda_+\sigma x = 2H\lambda_- x$. This, $\lambda_\pm\Pi = \lambda_\pm$, the Symmetry Lemma 2.14.b and $\lambda_+\zeta = \mathrm{id}\,\pi$ imply that $H\lambda_-\widehat{x} = H\lambda_- x + H\lambda_- x - 2H\lambda_- x = 0$. $\square$

Proof of the Calculation Lemma 2.15 is based on the following lemmas.

For a link denote by # the embedded connected sum of its components (they are not necessarily contained in disjoint cubes).

Lemma 2.16 (Connected Sum Lemma)**.** *The 'connected sum' map*

$$\# : E^{6k}(T^{0,4k-1}) \to E^{6k}(S^{4k-1}) \quad \textit{is} \quad \# = r_+ + r_- + \frac{H\lambda_+ + H\lambda_-}{2}.$$

Corollary 2.17. *For $k = 1, 3, 7$ let $\eta : S^{4k-1} \to S^{2k}$ be the Hopf map. The embedded connected sum $\#\zeta\eta$ of the components of $\zeta\eta$ is the generator of $E^{6k}(S^{4k-1})$.*

Proof. This follows from $r_\pm\zeta = 0$, $\lambda_+\zeta\eta = \eta$, $H\eta = 1$ and the Symmetry Lemma 2.14.b. $\square$

Proof of the Connected Sum Lemma 2.16. Clearly,

- $\#, r_+, r_-, \lambda_+, \lambda_-$ are homomorphisms.
- $\#$ is invariant under exchange of the components.
- $\# = r_+ + r_-$ for links with components contained in disjoint cubes.

So the lemma follows by the Whitehead Link Lemma 2.18 below because $Hw = 2$. $\square$

Lemma 2.18 (Whitehead Link Lemma)**.** *There is $\omega \in E^{6k}(T^{0,4k-1})$ such that $r_+\omega = r_-\omega = 0$, $\lambda_+\omega = 0$, $\lambda_-\omega = w$ and $\#\omega = 1$.*

Proof. (This proof is essentially known.) Denote coordinates in $\mathbb{R}^{6k}$ by $(x, y, z) = (x_1, \dots, x_{2k}, y_1, \dots, y_{2k}, z_1, \dots, z_{2k})$. The Borromean rings is the linking $S^{4k-1}_x \sqcup S^{4k-1}_y \sqcup S^{4k-1}_z \subset \mathbb{R}^{6k}$ of the three spheres given by the equations

$$\begin{cases} x = 0 \\ y^2 + 2z^2 = 1 \end{cases}, \qquad \begin{cases} y = 0 \\ z^2 + 2x^2 = 1 \end{cases} \qquad \text{and} \qquad \begin{cases} z = 0 \\ x^2 + 2y^2 = 1 \end{cases},$$

respectively, with natural orientations. Let $\omega : S^{4k-1} \sqcup S^{4k-1} \to \mathbb{R}^{6k}$ be the embedding whose first component is the connected sum of S^{4k-1}_x and S^{4k-1}_y, and the second component is S^{4k-1}_z.

By [Ha62] $\#\omega = 1$. Since each of the spheres S^{4k-1}_x, S^{4k-1}_y, S^{4k-1}_z spans an embedded disk in $\mathbb{R}^{6k}$, and the disks for S^{4k-1}_x and S^{4k-1}_y are disjoint, we have $r_\pm\omega = 0$. Since S^{4k-1}_x and S^{4k-1}_y span other disks disjoint from S_z, we have $\lambda_+\omega = 0$.

The spheres S^{4k-1}_x and S^{4k-1}_y span another disks whose intersections with S^{4k-1}_z form the Hopf link. Moreover, there are framings on the disks that are compatible with the orientations and give the standard framing on the Hopf link. Hence $\lambda_-\omega = w$. $\square$

Proof of the Calculation Lemma 2.15. Clearly, $\lambda_-\sigma = \lambda_-$.

By the Connected Sum Lemma 2.16 $r_-\sigma = \# = r_- + \frac{H\lambda_+ + H\lambda_-}{2}$.

We have $\sigma s_- = s_-\sigma$. So for calculation of $H\lambda_+\sigma[f]$ we may assume that $r_-[f] = 0$. Let $\psi_+ : E^{6k}(T^{0,4k-1}) \to E^{6k}(T^{0,4k-1})$ be the 'change of the orientation of the first component' map. Then omitting the argument $[f]$ we have

$$0 = r_- \overset{(1)}{=} \#\psi_+\sigma \overset{(2)}{=} (2r_- + H\lambda_+ + H\lambda_-)\psi_+\sigma \overset{(3)}{=} (2r_- - H\lambda_+ + H\lambda_-)\sigma \overset{(4)}{=}$$

$$= 2\# - H\lambda_+\sigma + H\lambda_- \overset{(5)}{=} H\lambda_+ + H\lambda_- - H\lambda_+\sigma + H\lambda_- \quad \Rightarrow \quad H\lambda_+\sigma = H\lambda_+ + 2H\lambda_-.$$

Here

(1) holds because two copies of the first component having opposite orientations 'cancel',

(2) and (5) hold by the Connected Sum Lemma 2.16 because $r_+\sigma = 0$, so $r_+\psi_+\sigma = 0$,

(3) holds because $r_-\psi_+ = r_-$, $\lambda_+\psi_+ = -\lambda_+$ and, analogously to the proof of the Symmetry Lemma 2.14.b, $H\lambda_-\psi_+ = H((-\iota_{2k}) \circ \lambda_-) = H\lambda_-$. [16]

(4) holds because $r_-\sigma = \#$ and $\lambda_-\sigma = \lambda_-$. $\square$

For $k = 1$ the two sketches below are complete proofs, because 'analogously to [Wa66, §4]' and 'by the analogue of [Wa66, Theorem 4]' can be replaced by 'by [Wa66, §4]' and 'by [Wa66, Theorem 4]'.

Sketch of an alternative proof of the Calculation Lemma 2.15. Take a representative $f : T^{0,4k-1} \to S^{6k}$ of an element from $E_0^{6k}(T^{0,4k-1}) = \ker r_+$. Denote by g a representative of $\sigma[f]$.

Analogously to [Wa66, §4] there is a unique normal framing of f such that $p_k(M_f) = 0$ for the $6k$-manifold M_f obtained from S^{6k} by surgery along f with this framing. Denote by $f_\pm \in H_{4k}(M_f)$ 'the homology classes of handles'. Analogously to [Wa66, Theorem 4], [Sk08'] $H\lambda_\pm[f] = f_\pm f_\mp^2$ and $6r_\pm[f] = f_\pm^3$. There is 'sliding handles' diffeomorphism $M_f \to M_g$. Under this diffeomorphism g_+, g_- go to $f_+, f_+ + f_-$. Since $r_+[f] = 0$, we obtain the required relations. $\square$

Sketch of an alternative proof of the Connected Sum Lemma 2.16. Take a representative $f : T^{0,4k-1} \to S^{6k}$ of an element from $E^{6k}(T^{0,4k-1})$. Denote by g a representative of $\#[f]$. The lemma follows by the analogue of [Wa66, Theorem 4] because 'the homology class of handle' $g_0 \in H_{4k}(M_g)$ 'goes to' $f_+ + f_-$, so $[g] = (f_+ + f_-)^3/6$. $\square$

3 Proofs of Lemmas

3.1 Proof of the Standardization Lemma 2.1

Proof of (a) for $X = D^p_+$. Take an embedding $g : T^{p,q}_+ \to S^m$. Since every two embeddings of a disk into S^m are isotopic, we can make an isotopy of S^m and assume that $g = \mathfrak{i}$ on $D^p_+ \times D^q_-$.

The ball D^m_- is contained in a tubular neighborhood of $\mathfrak{i}(D^p_+ \times D^q_-)$ in S^m relative to $\mathfrak{i}(D^p_+ \times \partial D^q_-)$. The image $g(D^p_+ \times \operatorname{Int} D^q_+)$ is disjoint from some tighter such tubular neighborhood. Hence by the Uniqueness of Tubular Neighborhood Theorem we can make an isotopy of S^m and assume that $g(D^p_+ \times \operatorname{Int} D^q_+) \cap D^m_- = \emptyset$. Then g is standardized. $\square$

Proof of (b) for $X = D^p_+$. Take an isotopy g between standardized embeddings $T^{p,q}_+ \to S^m$. By the 1-parametric version of 'every two embeddings of a disk into S^m are isotopic' we can make a self-isotopy of $\operatorname{id} S^m$, i.e. a level-preserving autodiffeomorphism of $S^m \times I$ identical on $S^m \times \{0,1\}$, and assume that $g = \mathfrak{i} \times \operatorname{id} I$ on $D^p_+ \times D^q_- \times I$.

The ball $D^m_- \times I$ is contained in a tubular neighborhood V of $\mathfrak{i}(D^p_+ \times D^q_-) \times I$ in $S^m \times I$ relative to $\mathfrak{i}(D^p_+ \times \partial D^q_-) \times I$. We may assume that $V \cap S^m \times k$ is 'almost D^m_-' for each $k = 0, 1$.

The image $g(D^p_+ \times \operatorname{Int} D^q_+ \times I)$ is disjoint from some tighter such tubular neighborhood, whose intersection with $S^m \times k$ is $V \cap S^m \times k$ for each $k = 0, 1$. Hence by the Uniqueness of Tubular Neighborhood Theorem we can make an isotopy of $S^m \times I$ relative to $S^m \times \{0,1\}$ and assume that $g(D^p_+ \times \operatorname{Int} D^q_+ \times I) \cap D^m_- \times I = \emptyset$. Then g is standardized. $\square$

Extend $\mathfrak{i}$ to $\sqrt{2}D^{p+1} \times D^{q+1}$ by the same formula as in the definition of $\mathfrak{i}$. For $\gamma \leq \sqrt{2}$ denote $\Delta_\gamma := \mathfrak{i}(\gamma D^{p+1} \times \{-1_q\}) \subset \operatorname{Int} D^m_-$.

[16] I conjecture that $r_+\psi_+ = -r_+$.

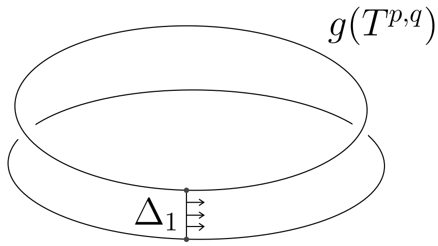

Figure 2: To the proof of the Standardization Lemma 2.1.a for $X = S^p$

Proof of (a) for $X = S^p$. Take an embedding $g : T^{p,q} \to S^m$. Since $m > 2p + q$, every two embeddings $S^p \times D^q \to S^m$ are isotopic (this is a trivial case of Theorem 1.10). So we can make an isotopy and assume that $g = \mathrm{i}$ on $S^p \times D^q_-$.

Since $m > 2p + q + 1$, by general position we may assume that $\operatorname{im} g \cap \Delta_1 = \partial\Delta_1$. Then there is γ slightly greater than 1 such that $\operatorname{im} g \cap \Delta_\gamma = \partial\Delta_1$. Take the 'standard' q-framing on Δ_γ tangent to $\mathrm{i}(\gamma D^{p+1} \times S^q)$ whose restriction to $\partial\Delta_1$ is the 'standard' normal q-framing of $\partial\Delta_1$ in $\operatorname{im} g$. Then the 'standard' $(m - p - q - 1)$-framing normal to $\mathrm{i}(\gamma D^{p+1} \times S^q)$ is an $(m - p - q - 1)$-framing on $\partial\Delta_1$ normal to $\operatorname{im} g$. Using these framings we construct

• an orientation-preserving embedding $H : D^m_- \to D^m_-$ onto a tight neighborhood of Δ_1 in D^m_-, and

• an isotopy h_t of $\operatorname{id} T^{p,q}$ shrinking $S^p \times D^q_-$ to a tight neighborhood of $S^p \times \{-1_q\}$ in $S^p \times D^q_-$ such that

$$H(\Delta_{\sqrt{2}}) = \Delta_\gamma, \quad H\,\mathrm{i}(S^p \times D^q_-) = H(D^m_-) \cap \operatorname{im} g \quad \text{and} \quad H\,\mathrm{i} = \mathrm{i}\,h_1 \quad \text{on} \quad S^p \times D^q_-.$$

Embedding H is isotopic to $\operatorname{id} D^m_-$ by [Hi76, Theorem 3.2]. This isotopy extends to an isotopy H_t of $\operatorname{id} S^m$ by the Isotopy Extension Theorem [Hi76, Theorem 1.3]. Then $H_t^{-1} g h_t$ is an isotopy of g. Let us prove that embedding $H_1^{-1} g h_1$ is standardized.

We have $H_1^{-1} g h_1 = H_1^{-1}\,\mathrm{i}\,h_1 = \mathrm{i}$ on $S^p \times D^q_-$. Also if $H_1^{-1} g h_1(S^p \times \operatorname{Int} D^q_+) \not\subset \operatorname{Int} D^m_+$, then there is $x \in S^p \times \operatorname{Int} D^q_+$ such that $g h_1(x) \in H(D^m_-)$. Then $g h_1(x) = H\,\mathrm{i}(y) = \mathrm{i}\,h_1(y) = g h_1(y)$ for some $y \in S^p \times D^q_-$. This contradicts to the fact that $g h_1$ is an embedding. $\square$

An embedding $F : N \times I \to S^m \times I$ is a *concordance* if $N \times k = F^{-1}(S^m \times k)$ for each $k = 0, 1$. Embeddings are called *concordant* if there is a concordance between them.

Proof of (b) for $X = S^p$. Take an isotopy g between standardized embeddings. The restriction $g|_{S^p \times D^q_-}$ is an isotopy between standard embeddings. So this restriction gives an embedding $g' : S^p \times D^q_- \times S^1 \to S^m \times S^1$ homotopic to $\mathrm{i}\,|_{S^p \times 0} \times \operatorname{id} S^1$. Since $m + 1 > 2(p + 1)$, by general position $g'|_{S^p \times 0 \times S^1}$ is isotopic to $\mathrm{i}\,|_{S^p \times 0} \times \operatorname{id} S^1$. Since $m > 2p + q + 1$, the Stiefel manifold $V_{m-p,q}$ is $(p+1)$-connected. Hence every two maps $S^1 \times S^p \to V_{m-p,q}$ are homotopic. Therefore g' is isotopic to $\mathrm{i} \times \operatorname{id} S^1$. So we can make a self-isotopy of $\operatorname{id} S^m$, i.e. a level-preserving autodiffeomorphism of $S^m \times I$ identical on $S^m \times \{0, 1\}$, and assume that $g = \mathrm{i} \times \operatorname{id} I$ on $S^p \times D^q_- \times I$.

Since $m > 2p + q + 2$, by general position we may assume that $\operatorname{im} g \cap \Delta_1 \times I = \partial\Delta_1 \times I$. Then there is a disk $\Delta \subset D^m_- \times I$ such that

$$\operatorname{Int} \Delta \supset \Delta_1 \times (0, 1), \quad \Delta \cap D^m_- \times \{0, 1\} = \Delta_{\sqrt{2}} \times \{0, 1\} \quad \text{and} \quad \operatorname{im} g \cap \Delta = \partial\Delta_1 \times I.$$

Take the 'standard' q-framing on Δ tangent to $\mathrm{i}(\sqrt{2} D^{p+1} \times S^q) \times I$ whose restriction to $\partial\Delta_1 \times I$ is the 'standard' normal q-framing of $\partial\Delta_1 \times I$ in $\operatorname{im} g$. Then the 'standard' $(m - p - q - 1)$-framing on $\partial\Delta_1 \times I$ normal to $\mathrm{i}(\sqrt{2} D^{p+1} \times S^q) \times I$ is an $(m - p - q - 1)$-framing on $\partial\Delta_1 \times I$ normal to $\operatorname{im} g$. Using these framings we construct

• an orientation-preserving embedding $H : D^m_- \times I \to D^m_- \times I$ onto a neighborhood of $\Delta_1 \times I$ in $D^m_- \times I$, and

• an isotopy h_t of $\mathrm{id}\, T^{p,q} \times I$ shrinking $S^p \times D^q_- \times I$ to a neighborhood of $S^p \times \{-1_q\} \times I$ in $S^p \times D^q_- \times I$ such that

$$H(\Delta_{\sqrt{2}} \times I) = \Delta, \quad H(\mathrm{i}(S^p \times D^q_-) \times I) = H(D^m_- \times I) \cap \mathrm{im}\, g$$

$$\text{and} \quad H \circ (\mathrm{i} \times \mathrm{id}\, I) = (\mathrm{i} \times \mathrm{id}\, I) \circ h_1 \quad \text{on} \quad S^p \times D^q_- \times I.$$

Analogously to the proof of (a) embedding H is isotopic to $\mathrm{id}(D^m_- \times I)$, such an isotopy extends to an isotopy H_t of $\mathrm{id}(S^m \times I)$, and $H_t^{-1} g h_t$ is an isotopy from g to a standardized isotopy $H_1^{-1} g h_1$. $\square$

3.2 Proof of the Group Structure Lemma 2.2

Let us prove that *the sum is well-defined*, i.e. that for standardized embeddings $f, g : X \times S^q \to S^m$ the isotopy class of $h_{f,g}$ depends only on $[f]$ and $[g]$. For this let us define parametric connected sum of isotopies. Take isotopic standardized embeddings f, f' and g, g'. By the Standardization Lemma 2.1.b there are standardized isotopies $F, G : X \times S^q \times I \to S^m \times I$ between f and f', g and g'. Define an isotopy

$$H : X \times S^q \times I \to S^m \times I \quad \text{by} \quad H(x, y, t) = \begin{cases} F(x, y, t) & y \in D^q_+ \\ R(G(x, Ry, t)) & y \in D^q_- \end{cases}.$$

Then H is an isotopy between h_{fg} and $h_{f'g'}$.

(The isotopy class of H may depend on F, G not only on their isotopy classes.)

Clearly, $\mathrm{i} : X \times S^q \to S^m$ *represents the zero element.*

Denote by R^t be the rotation of $\mathbb{R}^s = \mathbb{R}^2 \times \mathbb{R}^{s-2}$ whose restriction to the plane $\mathbb{R}^2 \times 0$ is the rotation through the angle $+\pi t$ and which leaves the orthogonal complement fixed.

Let us prove *the commutativity.* Each embedding $f : T^{p,q} \to \mathbb{R}^m$ is isotopic to $R^{-t} \circ f \circ (\mathrm{id}\, S^p \times R^t)$. Hence the embedding h_{fg} is isotopic to $R^1 \circ h_{fg} \circ (\mathrm{id}\, S^p \times R^1) = h_{gf}$.

Let us prove *the associativity.* Define $D^m_{++} \subset S^m$ by equations $x_1 \geq 0$ and $x_2 \geq 0$. Define $D^m_{+-}, D^m_{-+}, D^m_{--}$ analogously. Analogously to (or by) the Standardization Lemma 2.1.a each element in $E^m(X \times S^q)$ has a representative f such that

$$f(S^p \times D^q_{++}) \subset D^m_{++} \quad \text{and} \quad f|_{S^p \times (D^q - D^q_{++})} = \mathrm{i}.$$

Let $f, g, s : X \times S^q \to S^m$ be such representatives of three elements of $E^m(X \times S^q)$. Then both $[f] + ([g] + [s])$ and $([f] + [g]) + [s]$ have a representative defined by

$$h_{fgs}(x, y) := \begin{cases} f(x, y) & y \in D^q_{++} \\ R_{1/2}(g(x, R_{1/2} y)) & y \in D^q_{+-} \\ R_1(s(x, R_1 y)) & y \in D^q_{--} \\ \mathrm{i}(x, y) & y \in D^q_{-+} \end{cases}.$$

Let us prove that $[\overline{f}] + [f] = 0$. Clearly, $\overline{f}$ is standardized. Embedding $h_{f\overline{f}}$ can be extended to an embedding $X \times D^{q+1} \to D^{m+1}$ as follows. (Analogous minor modification should be done in [Ha66, 1.6] because the embedding $D^{n+1} \to D^{n+q+1}$ constructed there is not orthogonal to the boundary.) Represent an element of D^u as (a_0, a), where $a_0 \in [-1, 1]$ and $a \in \sqrt{1 - a_0^2} D^{u-1}$. Define

$$\gamma : D^u \to D^u \quad \text{by} \quad \gamma(a_0, a) := \left(\frac{a_0(1 + |a|^2)}{1 + \sqrt{1 - a_0^2 - a_0^2 |a|^2}}, a \right) \quad \text{and}$$

$$H : X \times D^{q+1} \to D^{m+1} \quad \text{by} \quad H(x, \gamma(y)) := \gamma(f(x,y)).$$

Since $RR_1 = R_2$, the map H is well-defined. Using $\mathrm{pr}_1\,\gamma(a_0, a) = \frac{1}{a_0} - \sqrt{\frac{1-a_0^2}{a_0^2} - |a|^2}$ for $a_0 \neq 0$, one can check that H is a smooth embedding (in particular, orthogonal to the boundary). Hence embedding $h_{f\overline{f}}$ is isotopic to i by the following Triviality Lemma 3.1. $\square$

Lemma 3.1 (Triviality Lemma)**.** *Let X denote either D^p_+ or S^p. Assume that $m \geq 2p+q+3$ for $X = S^p$ and $m \geq q+3$ for $X = D^p_+$. An embedding $X \times S^q \to S^m$ is isotopic to* i *if and only if it extends to an embedding $X \times D^{q+1} \to D^{m+1}$.*

Proof. For each u observe that $D^u \cong \Delta^u := D^u \bigcup\limits_{S^{u-1}=S^{u-1}\times 0} S^{u-1} \times I$.

The 'only if' part is proved by 'capping' an isotopy $X \times S^q \times I \to S^m \times I$ to i, i.e. by taking its union with $\mathrm{i} : X \times D^{q+1} \to D^{m+1}$.

(The 'only if' part does not require the dimension assumption.)

Let us prove the 'if' part. Represent the extension as an embedding $f : X \times \Delta^{q+1} \to \Delta^{m+1}$ such that $f^{-1}(\partial\Delta^{m+1}) = X \times \partial\Delta^{q+1}$. Analogously to the Standardization Lemma 2.1.a f is isotopic relative to $X \times \partial\Delta^{q+1}$ to an embedding g such that

$$g(X \times S^q \times I) \subset S^m \times I \quad \text{and} \quad g = \mathrm{i} \quad \text{on} \quad X \times D^{q+1}.$$

Then the abbreviation $g : X \times S^q \times I \to S^m \times I$ of g is a concordance from given embedding $g_0 : X \times S^q \to S^m$ to i.

If $X = S^p$, then $m \geq p+q+3$. Hence concordant embeddings g_0 and i are isotopic by [Hu70, Corollary 1.4] for $Q = S^m$, $X_0 = Y = \emptyset$.

If $X = D^p_+$, then $m \geq q+3$. Hence the concordance $g : (1, 0_p) \times S^q \times I \to S^m \times I$ is isotopic to an isotopy relative to $(1, 0_p) \times S^q \times \{0, 1\}$ by [Hu70, Theorem 1.5] for $Q = S^m$, $X_0 = Y = \emptyset$. The latter isotopy has a normal p-framing that coincides with the p-framing on $(1, 0_p) \times S^q \times \{0, 1\}$ defined by $g_0|_{(1,0_p)\times S^q}$ and $\mathrm{i}\,|_{(1,0_p)\times S^q}$. The vectors at $(x, t) \in S^m \times I$ are tangent to $S^m \times \{t\}$. So the latter isotopy extends to an isotopy between g_0 and i. $\square$

Remark 3.2. (a) The dimension restriction in the Triviality Lemma 3.1 for $X = S^p$ could be relaxed to $m \geq \max\{2p+q+2, p+q+3\}$ (which is sharp by (b,c)). This is so by Remark 2.5.c and because we use analogue of the Standardization Lemma 2.1.a not 2.1.b.

(b) The analogue of the Triviality Lemma 3.1 $m = q+2$ is false (both for $X = D^p_+$ and for $X = S^p$). This follows because there exist concordant non-isotopic embeddings $S^q \to S^{q+2}$ with trivial normal bundle.

(c) The analogue of the Triviality Lemma 3.1 for $X = S^p$ and $m = 2p+q+1$ is false. Indeed, let $x \in \pi_p(V_{p+q+1,q+1})$ be a generator. Then embedding $\tau^m_{q,p}(x)|_{T^{q,p}} : T^{q,p} \to S^m$ is not isotopic to i but extends to an embedding $\tau^{m+1}_{q+1,p}(x)|_{T^{q+1,p}_+} : T^{q+1,p}_+ \to D^{m+1}_+$. The former follows because $x \neq 0$, so $\alpha(\tau^m_{q,p}(x)|_{T^{q,p}}) \neq \alpha(\mathrm{i})$ [Sk02, Torus Lemma 6.1]. The latter follows because the stabilization map $V_{p+q+1,q+1} \to V_{p+q+2,q+2}$ induces an epimorphism on π_p, hence $\tau^{m+1}_{q+1,p}(T^{q+1,p}_\pm) \subset D^{m+1}_\pm$ and $\tau^{m+1}_{q+1,p} = \mathrm{i}_{m,m+1} \circ \tau^m_{q,p}$ on $T^{q,p}$.

3.3 Proof of the Smoothing Lemma 1.1

Lemma 3.3 (see proof below)**.** *For $m \geq 2p+q+3$ there is a homomorphism*

$$\overline{\sigma} : E^m(T^{p,q}) \to E^m(S^{p+q}) \quad \textit{such that} \quad \overline{\sigma} \circ \mathrm{i}\# = \mathrm{id}\, E^m(S^{p+q}).$$

The Smoothing Lemma 1.1 follows because Lemma 3.3 and $q_\# \circ \mathrm{i}\# = 0$ imply that $q_\# \oplus \overline{\sigma}$ is an isomorphism.

Lemma 3.3 is known [CRS12, Proposition 5.6] except for the non-trivial assertion that $\overline{\sigma}$ is a homomorphism.

Proof of Lemma 3.3: definition of $\overline{\sigma}$ and proof that $\sigma \circ \mathrm{i}\# = \mathrm{id}\, E^m(S^{p+q})$. The map $\overline{\sigma}$ is 'embedded surgery of $S^p \times *$', cf. equivalent definition below. We give an alternative detailed construction following [CRS12, Proposition 5.6]. Take $f \in E^m(T^{p,q})$. By the Standardization Lemma 2.1.a there is a standardized representative $f' : T^{p,q} \to S^m$ of f. Identify

$$S^{p+q} \quad \text{and} \quad S^p \times D^q_+ \bigcup_{S^p \times \partial D^q_+} D^{p+1} \times \partial D^q_+$$

by a diffeomorphism. Define an embedding

$$i' : D^{p+1} \times \partial D^q_+ \to D^m_- \quad \text{by} \quad i'(x,(0,y)) := (-\sqrt{1-|x|^2}, y, 0_l, x)/\sqrt{2}.$$

Then i' is an extension of the abbreviation $S^p \times \partial D^q_+ \to \partial D^m_+$ of $\mathrm{i}_{m,p,q}$. Infinite derivative for $|x| = 1$ means that i' meets the boundary regularly. Hence i' and $f'|_{S^p \times D^q_+}$ form together a (C^1-smooth) embedding $g : S^{p+q} \to S^m$. Let $\overline{\sigma}(f) := [g]$.

The map $\overline{\sigma}$ is well-defined for $m \ge 2p+q+3$ by the Standardization Lemma 2.1.b because the above construction of $\overline{\sigma}$ has an analogue for isotopy, cf. §3.2.

Clearly, $\overline{\sigma} \circ \mathrm{i}\#(g) = \overline{\sigma}(0\#g) = \overline{\sigma}(0) + g = 0 + g = g$.

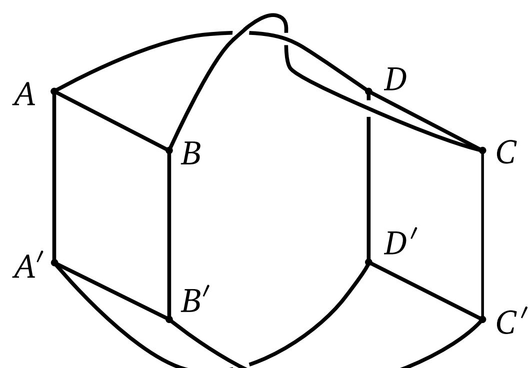

Figure 3: *To the proof that $\overline{\sigma}$ is a homomorphism.* This picture illustrates the proof by the case $p = 0$, $q = 1$ and $m = 3$ (which values are not within the dimension range $m \ge 2p+q+3$). The part above plane $ABCD$ stands for $\widehat{D^m_+}$. The part below plane $A'B'C'D'$ stands for $\widehat{D^m_-}$. The part between the planes stands for $S^{m-1} \times D^1$. The upper curved lines stand for $f_+(S^p \times S^{q-1}) = u(S^p \times S^{q-1})$. The bottom curved lines stand for $f_-(S^p \times S^{q-1}) = u(S^p \times S^{q-1})$. The union of segments $A'A, B'B, C'C$ and $D'D$ stands for $u(S^p \times S^{q-1} \times D^1)$. The union of segments $A'A$ and $B'B$ stands for $\mathrm{i}(S^p \times 1_{q-1}) \times D^1$. The quadrilateral $A'ABB'$ stands for the 'surgery disk' $\mathrm{i}(D^{p+1} \times D^{q-1}_+) \times D^1$. The union of the upper curved lines and the segment AB stands for the $(p+q)$-disk Δ_+. Analogously for Δ_-. The union of Δ_+, Δ_- and the segments $C'C$ and $D'D$ stands for the $(p+q)$-sphere that is the image of a representative of $\overline{\sigma}[u]$. The union of Δ_+ and CD stands for Σ_+. Analogously for Σ_-. The quadrilateral $C'CDD'$ stands for the tube $\mathrm{i}(D^{p+1} \times D^{q-1}_-) \times D^1$.

Proof of Lemma 3.3: beginning of the proof that $\overline{\sigma}$ is a homomorphism. For each n identify

$$S^n \quad \text{and} \quad \widehat{D^n_+} \bigcup_{\widehat{\partial D^n_+} = S^{n-1} \times 1} S^{n-1} \times D^1 \bigcup_{S^{n-1} \times \{-1\} = \widehat{\partial D^n_-}} \widehat{D^n_-}, \quad \text{where } \widehat{A} \text{ is a copy of } A.$$

Then $S^{n-1}=S^{n-1}\times 0\subset S^n$. Let $\mathrm{i}=\mathrm{i}_{m-1,p,q-1}$. Under the identifications $\widehat{\partial D^n_\pm}=S^{n-1}\times\{\pm1\}$, $n\in\{m,q\}$, the embedding $\mathrm{i}_{m,p,q}$ goes to $\mathrm{i}\,|_{S^p\times S^{q-1}}$. Hence analogously to (or by) the Standardization Lemma 2.1.a each element in $E^m(T^{p,q})$ has a representative f such that

- $f(S^p\times\widehat{D^q_+})\subset\widehat{D^m_+}$;
- $f=\mathrm{i}_{m,p,q}$ on $S^p\times\widehat{D^q_-}$ (the image of this embedding lyes in $\widehat{D^m_-}$);
- $f=\mathrm{i}\,|_{S^p\times S^{q-1}}\times\mathrm{id}\,D^1$ on $S^p\times S^{q-1}\times D^1$ (the image of this embedding lies in $S^{m-1}\times D^1$).

Take embeddings $f_\pm:T^{p,q}\to S^m$ satisfying the above properties. Then $[f_+]+[f_-]$ has a representative $u:T^{p,q}\to S^m$ such that

- $u=f_+$ on $S^p\times\widehat{D^q_+}$;
- $u=(\mathrm{id}\,S^p\times R)\circ f_-\circ(\mathrm{id}\,S^p\times R)$ on $S^p\times\widehat{D^q_-}$;
- $u=\mathrm{i}\,|_{S^p\times S^{q-1}}\times\mathrm{id}\,D^1$ on $S^p\times S^{q-1}\times D^1$.

For completion of the proof that $\overline{\sigma}$ is a homomorphism we need an equivalent definition of $\overline{\sigma}(f)$ (i.e. of the embedded surgery of $S^p\times 1_q$).

First we assume that $p=0$, i.e. define the embedded connected sum of embeddings $f_{-1},f_1:S^q\to S^m$ whose images are disjoint. Take an embedding $l:D^1\times D^q_-\to S^m$ such that

$$l=f_k\quad\text{on}\quad k\times D^q_-\quad\text{and}\quad l(D^1\times D^q_-)\cap f_k(S^q)=l(k\times D^q_-)\quad\text{for}\quad k=\pm1.$$

Define $h:S^q\to S^m$ by

$$h(x):=\begin{cases}f_0(x) & x\in\widehat{D^q_+}\\ l(x) & x\in D^1\times\partial D^q_+\\ f_1(x) & x\in\widehat{D^q_-}\end{cases}.$$

Then a representative of $[f_0]+[f_1]$ is obtained from h by smoothing of the 'dihedral corner' along $h(S^0\times\partial D^q_+)$. This smoothing is local replacement of embedded $(I\times 0\cup 0\times I)\times D^{q-1}$ by embedded $C\times D^{q-1}$, where $C\subset I^2$ is a smooth curve joining $(0,1)$ to $(1,0)$ and such that $C\cup[1,2]\times 0\cup 0\times[1,2]$ is smooth. This smoothing is 'canonical', i.e. does not depend on the choice of C. Cf. [Ha62, Proof of 3.3] and, for non-embedded version, [U].

Let us generalize this definition to arbitrary p. Take an embedding $l:D^{p+1}\times D^q_-\to S^m$ such that

$$l=f\quad\text{on}\quad S^p\times D^q_-\quad\text{and}\quad l(D^{p+1}\times D^q_-)\cap f(T^{p,q})=l(S^p\times D^q_-).$$

Define $h:S^{p+q}\to S^m$ by

$$h(x):=\begin{cases}f(x) & x\in S^p\times D^q_+\\ l(x) & x\in D^{p+1}\times\partial D^q_+\end{cases}.$$

Then a representative of $\overline{\sigma}(f)$ is obtained from h by 'canonical' smoothing of the 'dihedral corner' along $h(S^p\times\partial D^q_+)$ analogous to the above case $p=0$.

This definition is equivalent to that from the beginning of proof of Lemma 3.3 because there are a closed ε-neighborhood U of the image of l (for some small $\varepsilon>0$) and a self-diffeomorphism $G:S^m\to S^m$ such that $G(D^m_-,\mathrm{i}(T^{p,q}_-),i'(D^{p+1}\times\partial D^q_+))=(U,U\cap f(T^{p,q}),U\cap h(S^{p+q}))$.

The result of the above surgery does not depend on the choices involved because $\overline{\sigma}(f)$ is well-defined.

Completion of the proof that $\overline{\sigma}$ is a homomorphism. [17] (This argument appeared after a discussion with A. Zhubr.) Recall that a representative of $\overline{\sigma}[u]$ is obtained from u by 'embedded

[17] Proof of Lemma 3.3 from the previous version of this paper, and so the 'second completion of the proof that $\overline{\sigma}$ is a homomorphism' there, are incorrect. The mistake is that $H^{p+q}(X,\partial;E^m(S^{p+q}))\cong H_0(X;E^m(S^{p+q}))\cong E^m(S^{p+q})$ could be non-zero for a $(p+q)$-manifold X.

surgery of $\mathrm{i}(S^p \times 1_{q-1}) \times 0$'. Recall that the isotopy class of an embedding $g : S^{p+q} \to S^m$ is defined by the image of g and an orientation on the image.

Denote

$$\Delta_\pm \ := \ u(S^p \times D^q_\pm) \ \cup \ \mathrm{i}(D^{p+1} \times D^{q-1}_+) \times \{\pm 1\} \ \cong \ S^p \times D^q \cup D^{p+1} \times D^{q-1}_+ \ \underset{PL}{\cong} \ D^{p+q}.$$

Then the oriented image of the representative of $\overline{\sigma}[u]$ is obtained by 'canonical' smoothing of corners from

$$\big(u(T^{p,q}) - \mathrm{i}(S^p \times D^{q-1}_+) \times D^1\big) \cup (\mathrm{i} \times \mathrm{id}\, D^1)\, \big(D^{p+1} \times \partial(D^{q-1}_+ \times D^1)\big) =$$

$$= \Delta_- \cup \mathrm{i}\, \partial(D^{p+1} \times D^{q-1}_-) \times D^1 \cup \Delta_+ \ \underset{PL}{\cong} \ D^{p+q} \times 0 \cup S^{p+q-1} \times I \cup D^{p+q} \times 1 \ \underset{PL}{\cong} \ S^{p+q}.$$

This oriented $(p+q)$-sphere is a connected sum of oriented $(p+q)$-spheres

$$\Sigma_\pm := \Delta_\pm \cup \mathrm{i}(D^{p+1} \times D^{q-1}_-) \times \{\pm 1\} \ \underset{PL}{\cong} \ 0 \times D^{p+q} \cup D^{p+q}_+ \ \underset{PL}{\cong} \ S^{p+q}$$

along the tube $\mathrm{i}(D^{p+1} \times D^{q-1}_-) \times D^1$. The image of a representative of $\overline{\sigma}[f_\pm]$ is obtained from $\Sigma_\pm$ by 'canonical' smoothing of the 'dihedral corner'. The corners of the tube $\mathrm{i}(D^{p+1} \times D^{q-1}_-) \times D^1$ can be 'canonically' smoothed to obtain an embedding $D^{p+q} \times D^1 \to S^m$. Thus $\overline{\sigma}[u] = \overline{\sigma}[f_+] + \overline{\sigma}[f_-]$. $\square$

3.4 Proof of Lemma 2.6

Consider the following diagram.

$$\begin{tikzcd}
E^{m+1}(T^{p,q+1}_+) \arrow[r, squiggly, "\overline{\rho}"] \arrow[d, dotted, "\lambda'"] & E^{m+1}(S^{q+1}) \arrow[d, "\xi"] \arrow[dr, squiggly, "\overline{\xi}"] & & & \\
\pi_q(S^l) \arrow[r, dashed, "\mu''"] \arrow[dr, dotted, "\mu'"] & \pi_q(V_{m-q,p+1}) \arrow[r, dashed, "\nu''"] \arrow[d, "\tau"] & \pi_q(V_{m-q,p}) \arrow[d, squiggly, "\overline{\tau}"] \arrow[dr, dashed, "\lambda''"] & & \\
 & E^m(T^{p+1,q}_+) \arrow[r, dotted, "\nu'"] \arrow[dr, "\rho"] & E^m(T^{p,q}_+) \arrow[r, dotted, "\lambda'"] \arrow[d, squiggly, "\overline{\rho}"] & \pi_{q-1}(S^l) \arrow[d, dashed, "\mu''"] \arrow[dr, dotted, "\mu'"] & \\
 & & E^m(S^q) \arrow[r, "\xi"] \arrow[dr, squiggly, "\overline{\xi}"] & \pi_{q-1}(V_{m-q,p+1}) \arrow[r, "\tau"] \arrow[d, dashed, "\nu''"] & E^{m-1}(T^{p+1,q-1}_+) \arrow[d, dotted, "\nu'"] \\
 & & & \pi_{q-1}(V_{m-q,p}) \arrow[r, squiggly, "\overline{\tau}"] & E^{m-1}(T^{p,q-1}_+)
\end{tikzcd}.$$

Here

• the $\mu''\nu''\lambda''$ sequence is the exact sequence of the 'forgetting the last vector' bundle $S^l \to V_{m-q,p+1} \to V_{m-q,p}$;

• the exact $\tau\rho\xi$- and $\overline{\tau}\overline{\rho}\overline{\xi}$-sequences are defined in Theorem 1.10.

Let us prove the commutativity.

Let us prove that $\xi\overline{\rho} = \mu''\lambda'$ for the left upper square. By the Standardization Lemma 2.1.a each element of $E^{m+1}(T^{p,q+1}_+)$ is representable by a standardized embedding $f : D^p_+ \times S^{q+1} \to S^{m+1}$. Since $f|_{D^p_+ \times D^{q+1}_-} = \mathrm{i}$, there is a normal $(m-q)$-framing of $f(D^{q+1}_-)$ extending $f|_{D^p_+ \times D^{q+1}_-}$ and and a normal $(p+1)$-framing of $f(D^{q+1}_+)$ extending $f|_{D^p_+ \times D^{q+1}_+}$. Then $\xi[f|_{0_p \times S^q}] = \mu''\lambda'[f]$ by definitions of λ' and ξ presented in §2.2 and §1.3, respectively.

Relation $\lambda'' = \overline{\tau}\lambda'$ follows by definitions of λ' (§2.2) and of λ''.

(Recall definition of λ''. A map $\mathbb{R}^k \times X \to \mathbb{R}^n$ is called a *linear monomorphism* if its restriction to $\mathbb{R}^k \times x$ is a linear monomorphism for each $x \in X$. Represent an element $x \in \pi_q(V_{m-q,p})$ by a linear monomorphism

$$f : \mathbb{R}^p \times S^q \to \mathbb{R}^{m-q} = \mathbb{R}^p \times \mathbb{R}^{l+1} \quad \text{such that} \quad f(x,y) = (x,0) \quad \text{for each} \quad y \in D^q_-.$$

By the Covering Homotopy Property for the 'forgetting last vector' bundle $V_{m-q,p+1} \to V_{m-q,p}$ the restriction $f|_{\mathbb{R}^p \times D^q_+}$ extends to a linear monomorphism $s : \mathbb{R}^p \times \mathbb{R} \times D^q_+ \to \mathbb{R}^{m-q} = \mathbb{R}^p \times \mathbb{R}^{l+1}$. For each $y \in S^{q-1}$ we have $s(\mathbb{R}^p \times 0 \times y) = \mathbb{R}^p \times 0$, so $s(0_p, 1, y) \notin \mathbb{R}^p \times 0$. Hence we can define a map

$$g : S^{q-1} \to S^l \quad \text{by} \quad g(y) := n\,\mathrm{pr}_2 s(0_p, 1, y), \quad \text{where} \quad n(z) := z/|z|.$$

Let $\lambda''(x)$ be the homotopy class of g.)

The commutativity of other squares and triangles is obvious.

Clearly, $\lambda'\nu' = 0$. So the exactness of the $\lambda'\mu'\nu'$ sequence follows by the Snake Lemma, cf. [Ha66, proof of (6.5)]. $\square$

3.5 Proof of Lemma 2.8

Proof of Lemma 2.8 except the 'moreover' part. Since $l_1 t_1 = \mathrm{id}\, B_1$, the map t_1 is injective and $B_1' = \mathrm{im}\, t_1 \oplus \ker l_1$ (the mutually inverse isomorphisms are given by $x \mapsto (t_1 l_1 x, x - t_1 l_1 x)$ and $(y,z) \mapsto y + z$). Therefore

$$\ker b_1' \cap \ker l_1 = \mathrm{im}\, a_1' \cap \ker l_1 \subset \mathrm{im}\, t_1 \cap \ker l_1 = 0.$$

Hence $b_1'|_{\ker l_1}$ is injective. Thus $b_1' \ker l_1 \cong \ker l_1$.

We have a diagram

$$\begin{tikzcd}
 & & C_1 \arrow[dr, "c_1"] & & \\
A_1 \arrow[r, "a_1''"'] & \mathrm{im}\, t_1 \arrow[ur, "b_1 t_1^{-1}"] \arrow[r, "\pi b_1'"'] & \dfrac{C_1'}{b_1' \ker l_1} \arrow[r, "c_1''"'] & A_0 \arrow[r, "a_0''"'] & \mathrm{im}\, t_0
\end{tikzcd}.$$

Here π is the quotient map, a_k'' is the abbreviation of a_k, c_1'' is well-defined by the formula $c_1''(x + b_1' \ker l_1) := c_1'(x)$ for each $x \in C_1'$ (because $c_1' b_1' = 0$). Since t_1 is injective, the map $b_1 t_1^{-1}$ in the diagram is well-defined.

Clearly, the $a_1''(\pi b_1')c_1'' a_0''$-sequence is exact. The exactness of the $a_1''(b_1 t_1^{-1})c_1 a_0''$-sequence follows from

$$\ker a_0'' = \ker a_0' = \ker a_0 \overset{(*)}{=} \mathrm{im}\, c_1, \quad \ker c_1 = \mathrm{im}\, b_1 = \mathrm{im}(b_1 t_1^{-1})$$
$$\text{and} \quad \ker(b_1 t_1^{-1}) = t_1 \ker b_1 = t_1\, \mathrm{im}\, a_1 = \mathrm{im}\, a_1' = \mathrm{im}\, a_1''.$$

Here (*) holds because t_0 is injective. So C_1 and $C_1'/b_1' \ker l_1$ have isomorphic subgroups ($\mathrm{im}\, b_1 \cong \mathrm{coker}\, a_1'' \cong \mathrm{im}\, \pi b_1'$) with isomorphic quotients ($\mathrm{im}\, c_1 = \ker a_0'' = \mathrm{im}\, c_1''$). $\square$

Proof of Lemma 2.8, the 'moreover' part. Let $B := t_1 \,\mathrm{coker}\, a_1$ and $A = \mathrm{im}\, c_1 = \mathrm{im}\, c_1'$. We have the following diagram:

$$\begin{tikzcd}
 & t_1^{-1}B \arrow[r, "b_1"] \arrow[d, "t_1' \oplus 0"] & C_1 \arrow[d, dotted, "r"'] \arrow[dr, "c_1"] & & \\
0 \arrow[ur] \arrow[r] & B \oplus \ker l_1 \arrow[r, "b_1'"'] & C_1' \arrow[r, "c_1'"'] & A \arrow[r] & 0
\end{tikzcd}.$$

Denote by T the torsion subgroup of C_1'. Since C_1 is finite, both A and $B \cong t_1^{-1}B$ are finite. Hence $b_1'B \subset T$. Since C_1 is finite, $rC_1 \subset T$, so $c_1'|_T$ is surjective. Thus the sequence $0 \to B \overset{b_1'}{\to} T \overset{c_1'|_T}{\to} A \to 0$ is exact. Therefore $|T| = |A| \cdot |B| = |C_1|$.

Taking quotients of $B \oplus \ker l_1$ by $\ker l_1$ and of C_1' by $H := b_1' \ker l_1$ we obtain an exact sequence showing that $|C_1'/H| = |A| \cdot |B| = |T|$. Since $b_1'|_{\ker l_1}$ is injective and $\ker l_1$ is free, H is a free. So $T \cap H = \emptyset$. Since C_1' is a finitely generated abelian group, we can identify C_1' and $F \oplus T$ by some isomorphism for some free abelian group F (note that possibly $H \neq F \oplus 0$). Since $|C_1'/H| = |T|$, the composition $H \overset{\subsetneq}{\to} C_1' \overset{\mathrm{pr}_1}{\to} F$ is an isomorphism. The composition of $C_1' \overset{\mathrm{pr}_1}{\to} F$ and the inverse to this isomorphism makes H a direct summand in C_1'. $\square$

References

[Ad04] C.C. Adams, *The Knot Book: An elementary introduction to the mathematical theory of knots,* 2004.

[Av16] S. Avvakumov, *The classification of certain linked 3-manifolds in 6-space,* Moscow Mathematical Journal, 16:1 (2016) 1-25. arXiv:1408.3918.

[BG71] J.C. Becker and H. H. Glover, *Note on the Embedding of Manifolds in Euclidean Space,* Proc. of the Amer. Math. Soc., 27:2 (1971) 405-410. doi:10.2307/2036329

[Bo71] J. Boechat, *Plongements differentiables de varietes de dimension $4k$ dans $\mathbb{R}^{6k+1}$,* Comment. Math. Helv. 46:2 (1971) 141–161.

[BZ03] G. Burde and H. Zieschang, *Knots*, 2003, Walter de Gruyter, Berlin, New York.

[CDM] S.Chmutov, S.Duzhin, J.Mostovoy. *Introduction to Vassiliev Knot Invariants,* Cambridge Univ. Press, 2012. `http://www.pdmi.ras.ru/~duzhin/papers/cdbook`

[CF63] R.H. Crowell and R.H. Fox, *Introduction to Knot Theory*, 1963, Springer-Verlag, Berlin-Heidelberg-New York.

[CFS] D. Crowley, S.C. Ferry, M. Skopenkov *The rational classification of links of codimension > 2,* Forum Math. 26:1 (2014), 239-269. arXiv:1106.1455

[CRS04] M. Cencelj, D. Repovš and A. Skopenkov, *On the Browder-Haefliger-Levine-Novikov embedding theorems,* Proc. of the Steklov Inst. Math., 247 (2004)

[CRS07] M. Cencelj, D. Repovš and M. Skopenkov, *Homotopy type of the complement of an immersion and classification of embeddings of tori,* Russian Math. Surveys, 62:5 (2007)

[CRS12] M. Cencelj, D. Repovš and M. Skopenkov, *Classification of knotted tori in the 2-metastable dimension,* Mat. Sbornik, 203:11 (2012), 1654-1681. arXiv:0811.2745.

[CS11] D. Crowley and A. Skopenkov, *A classification of smooth embeddings of 4-manifolds in 7-space, II*, Intern. J. Math., 22:6 (2011) 731-757, arXiv:0808.1795.

[CS] D. Crowley and A. Skopenkov, *Embeddings of non-simply connected 4-manifolds in 7-space*, I, II, III, preprints.

[GW99] T. Goodwillie and M. Weiss, *Embeddings from the point of view of immersion theory, II*, Geometry and Topology, 3 (1999) 103–118.

[Ha61] A. Haefliger, *Plongements differentiables de varietes dans varietes*, Comment. Math. Helv., 36 (1961), 47–82.

[Ha62] A. Haefliger, *Knotted* $(4k-1)$*-spheres in* $6k$*-space*, Ann. of Math. 75 (1962) 452–466.

[Ha62'] A. Haefliger, *Differentiable links*, Topology, 1 (1962) 241–244.

[Ha66] A. Haefliger, *Differentiable embeddings of* S^n *in* S^{n+q} *for* $q>2$*,* Ann. Math., Ser.3, 83 (1966) 402–436.

[Ha66'] A. Haefliger, *Enlacements de spheres en codimension superiure à 2,* Comment. Math. Helv. 41 (1966-67) 51–72.

[Ha67] A. Haefliger, *Lissage des immersions-I,* Topology, 6 (1967) 221–240.

[Hi76] M. W. Hirsch, *Differential Topology*, 1976, Springer-Verlag, New York.

[Hu70] J. F. P. Hudson, *Concordance, isotopy and diffeotopy,* Ann. of Math. 91:3 (1970) 425–448.

[Ir65] M. C. Irwin, *Embeddings of polyhedral manifolds*, Ann. of Math. (2) 82 (1965) 1–14.

[Ka87] L. Kauffman, *On Knots,* Princeton University Press, 1987.

[Ke59] M. Kervaire, *An interpretation of G. Whitehead's generalization of H. Hopf's invariant,* Ann. of Math. 62 (1959) 345–362.

[Ko88] U. Koschorke, *Link maps and the geometry of their invariants,* Manuscripta Math. 61:4 (1988) 383–415.

[LNS] L. A. Lucas, O.M. Netu and O. Saeki, *A generalization of Alexander's torus theorem to higher dimensiona and an unknotting theorem for* $S^p \times S^q$ *embedded in* S^{p+q+2}, Kobe J. Math, 13 (1996), 145-165.

[Ma80] R. Mandelbaum, *Four-Dimensional Topology: An introduction*, Bull. Amer. Math. Soc. (N.S.) 2 (1980) 1-159.

[MAH] `http://www.map.mpim-bonn.mpg.de/High_codimension_embeddings`, Manifold Atlas Project (unrefereed page).

[MAK] `http://www.map.mpim-bonn.mpg.de/Knotted_tori`, Manifold Atlas Project (unrefereed page).

[MAL] `http://www.map.mpim-bonn.mpg.de/High_codimension_links`, Manifold Atlas Project (unrefereed page).

[MAP] `http://www.map.mpim-bonn.mpg.de/Parametric_connected_sum`, Manifold Atlas Project (unrefereed page).

[MAW] `http://www.map.mpim-bonn.mpg.de/Embeddings_just_below_the_stable_range:_classification#The_Whitney_invariant`, Manifold Atlas Project (unrefereed page).

[Mi72] R. J. Milgram, *On the Haefliger knot groups,* Bull. of the Amer. Math. Soc. 78:5 (1972) 861–865.

[Pa56] G. Paechter, *On the groups* $\pi_r(V_{mn})$, I, Quart. J. Math. Oxford, Ser.2, 7:28 (1956) 249–265.

[Po85] M.M. Postnikov, *Homotopy theory of CW-complexes*, Nauka, Moscow, 1985 (in Russian).

[PS96] V.V. Prasolov and A.B. Sossinsky, *Knots, Links, Braids, and 3-manifolds.* Amer. Math. Soc. Publ., Providence, R.I., 1996. Russian version: `http://www.mccme.ru/prasolov`

[Re48] K. Reidemeister, *Knotentheorie,* Chelsea Publishing Co., 1948.

[Ro76] D. Rolfsen, *Knots and Links,* AMS Chelsea Publishing, 1976.

[Sc71] R. Schultz, *On the inertia groups of a product of spheres,* Trans. AMS, 156 (1971), 137–153.

[Sk02] A. Skopenkov,*On the Haefliger-Hirsch-Wu invariants for embeddings and immersions,* Comment. Math. Helv. 77 (2002), 78–124.

[Sk06] A. Skopenkov, *Classification of embeddings below the metastable dimension,* arxiv:math/0607422v2

[Sk07] A. Skopenkov, *A new invariant and parametric connected sum of embeddings,* Fund. Math. 197 (2007), 253–269. arxiv:math/0509621

[Sk08] A. Skopenkov, *Embedding and knotting of manifolds in Euclidean spaces,* in: Surveys in Contemporary Mathematics, Ed. N. Young and Y. Choi London Math. Soc. Lect. Notes, 347 (2008) 248–342. arxiv:math/0604045

[Sk08'] A. Skopenkov, *Classification of smooth embeddings of 3-manifolds in 6-space*, Math. Zeitschrift, 260:3, 2008, 647-672, arxiv:math/0603429

[Sk09] M. Skopenkov, *A formula for the group of links in the 2-metastable dimension,* Proc. AMS 137 (2009) 359–369. arxiv:math/0610320

[Sk10] A. Skopenkov, *A classification of smooth embeddings of 4-manifolds in 7-space*, I, Topol. Appl., 157 (2010) 2094-2110, arXiv:0808.1795.

[Sk10'] A. Skopenkov, *Embeddings of* k*-connected* n*-manifolds into* $\mathbb{R}^{2n-k-1}$, Proc. AMS, 138 (2010) 3377–3389, arXiv:0812.0263.

[Sk15] M. Skopenkov, *When is the set of embeddings finite?* Intern. J. Math. 26:7 (2015), arXiv:1106.1878.

[Sk] A. Skopenkov, *How do autodiffeomorphisms act on embeddings,* arXiv:1402.1853.

[U] `http://math.stackexchange.com/questions/368640/uniqueness-of-smoothed-corners`

[Wa66] C. T. C. Wall, *Classification Problems in Differential Topology. V. On Certain 6-Manifolds*, Invent. Math., **1** (1966) 355–374.

[Wa70] C. T. C. Wall, *Surgery on Compact Manifolds,* 1970, Academic Press, London.

[Zh16] A. Zhubr, *ON SMOOTHING EMBEDDINGS AND ISOTOPIES*, Math. Notes, 99 (2016), 946-947.